Liquid Neural Networks

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CSAIL MIT

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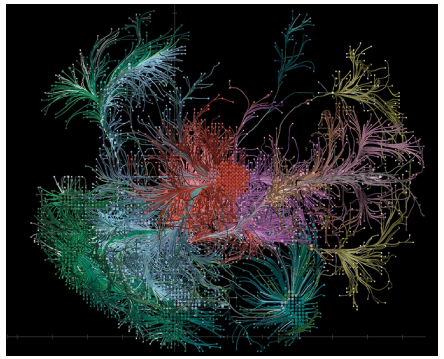
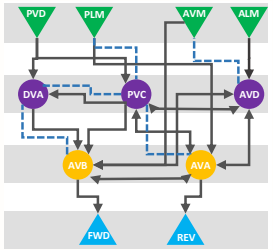
Nervous Systems
(Image: Allen Institute for Brain Science)



Neural Circuits



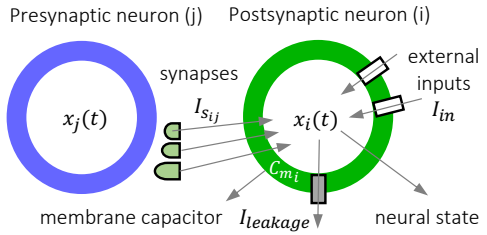
ICML 2020
ICRA 2019
NeurIPS Deep RL Symp 2017

Liquid Neural Networks

$$d\mathbf{x}(t)/dt = -\mathbf{x}(t)/\tau + \mathbf{S}(t)$$

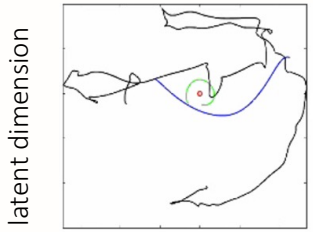
$$\mathbf{S}(t) = f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)(A - \mathbf{x}(t))$$

Neurons & Synapses



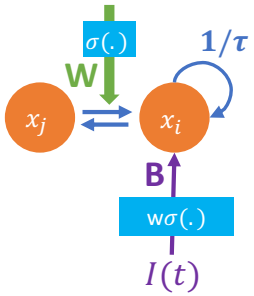
Nature Machine Intelligence 2020

Expressivity



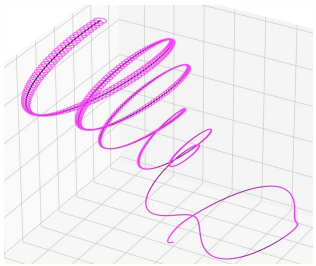
AAAI 2021

Causality



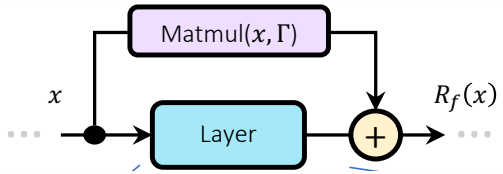
NeurIPS 2021

Robustness



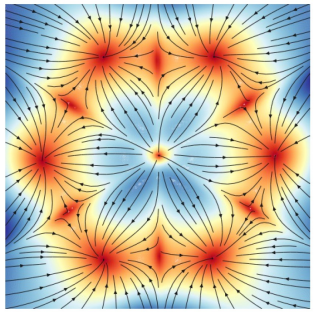
AAAI 2021 & 2022

memory



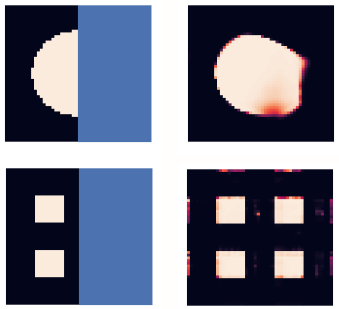
NeurIPS 2022, ICLR 2023
Nature MI 2022

Generative Modeling



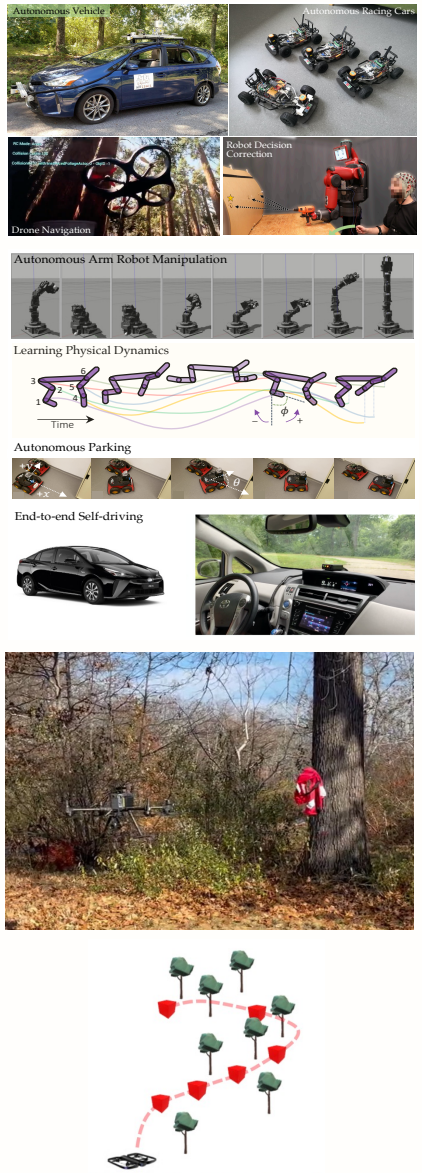
NeurIPS 2021

Extrapolation

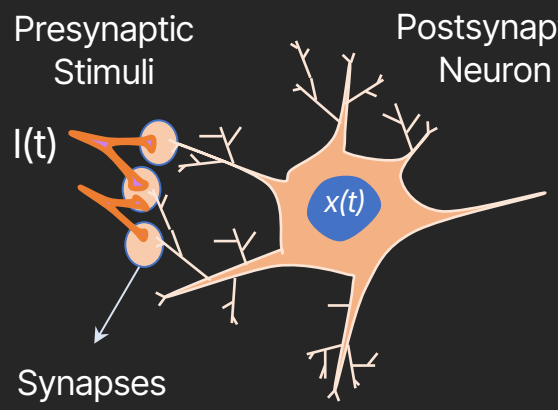


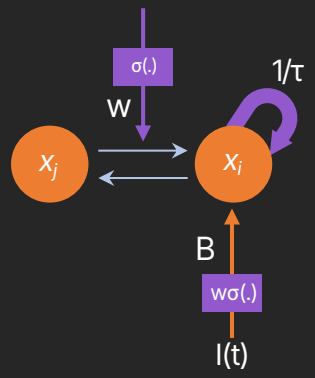
Science Robotics 2023

Mixed-horizon Decision-making



Brains

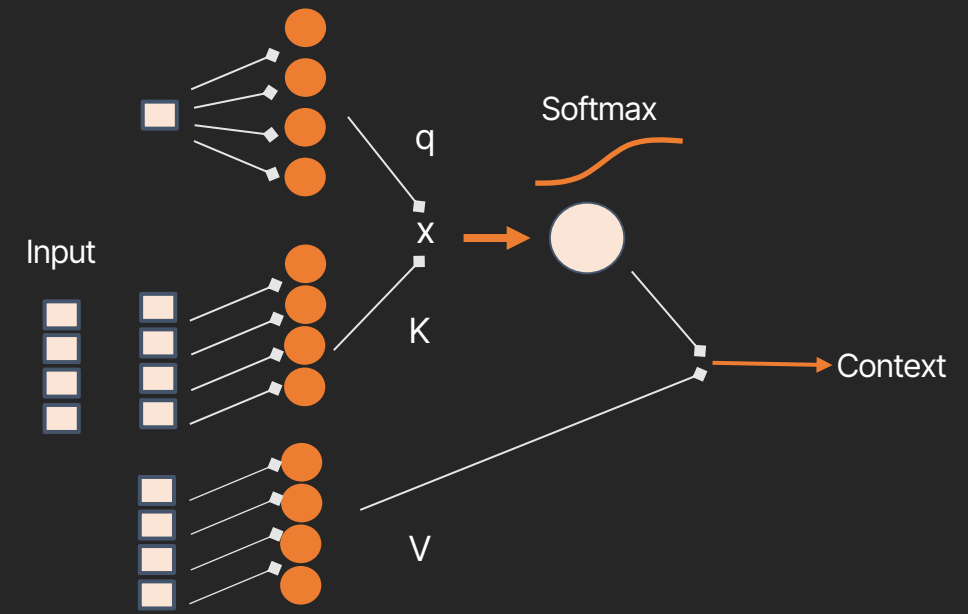


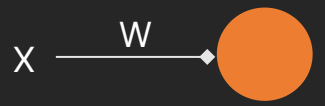


Main Operations

I. Scalar weighted operations
II. Threshold activations
III. Feedforward operations
IV. Multi-scale feedback
V. Probabilistic synapses
    I. Nonlinear functions
    II. Feedback system
    III. Information release pattern
VI. Nodes are continuous processes
VII. Governed by state-space models

Base of Gen AI: Transformers

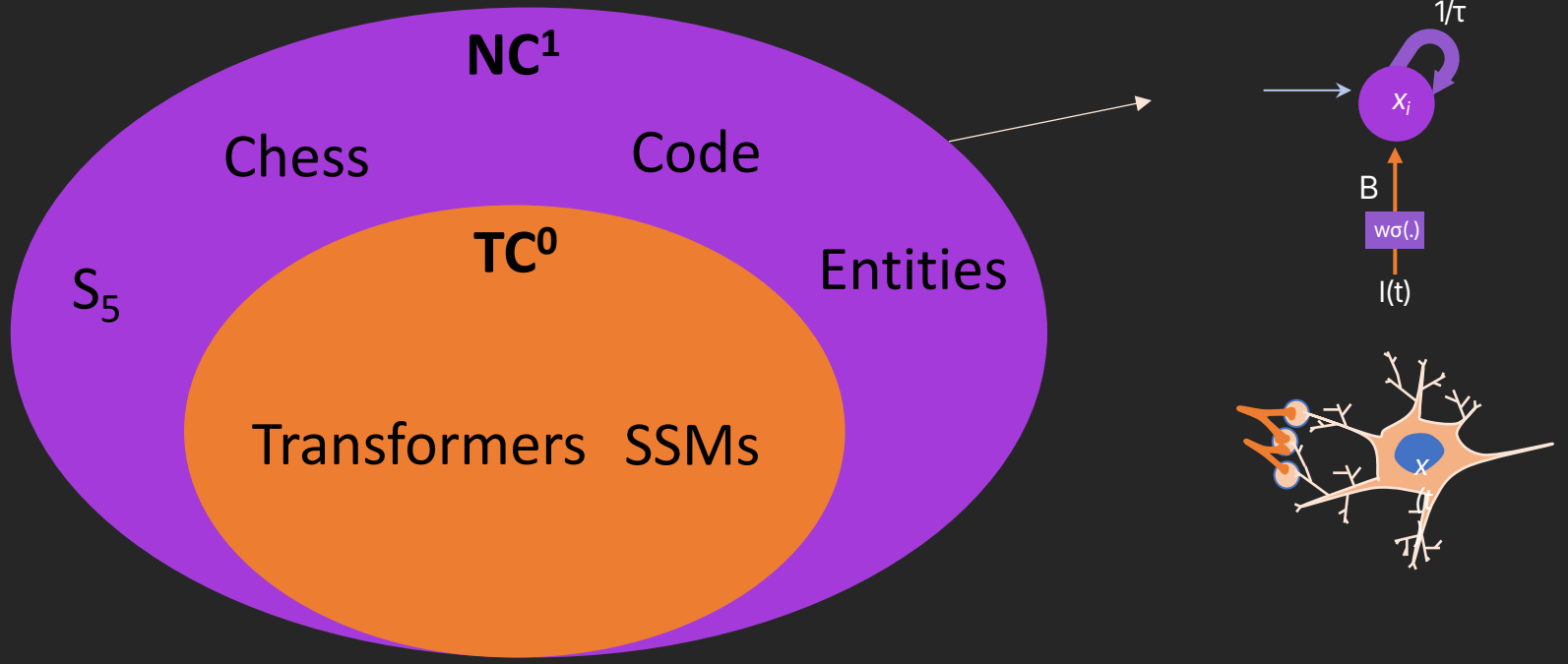




Main Operations

I. Scalar weighted operations
II. Threshold activations
III. Feedforward operations

Why bother?



NC (Nick's Class) = Problems that can be efficiently solved on a parallel Computer.
TC0 = Threshold Circuits of Depth 0
SSM = State-space model
RNN = recurrent neural network

Merrill et al. ICML 2024

Outline

- ✓ Liquid time-constant networks
- ✓ Autonomy with liquid networks
- ✓ Causality
- ✓ Closed-form Continuous-time decision-making
- ✓ Task understanding and out of distribution generalization

Outline

- ✓ **Liquid time-constant networks**
- ✓ Autonomy with liquid networks
- ✓ Causality
- ✓ Closed-form Continuous-time decision-making
- ✓ Task understanding and out of distribution generalization

Autonomy as a continuous-time dynamical system

- ✓ Practical decision making is naturally modeled a **continuous process**
- ✓ Dynamical system described by **differential equations**
- ✓ Recurrence, memory, and sparsity

What is a continuous- time/depth neural network?

$$\frac{dx}{dt} = f_{(n,k,l_{type})}(x(t), I(t), \theta)$$

Number of layers

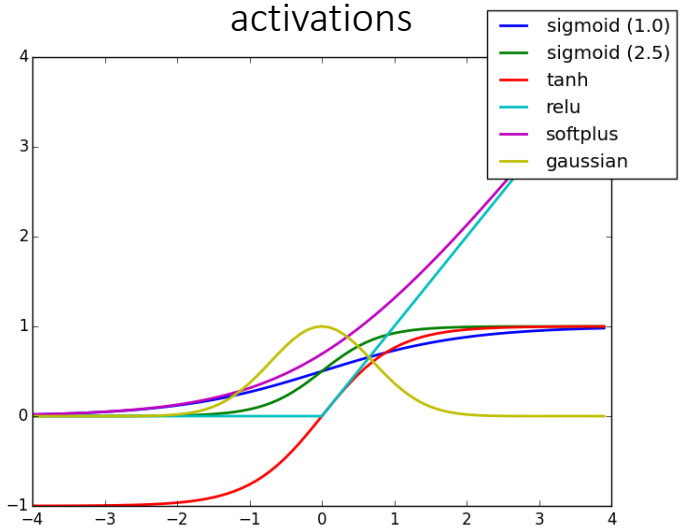
Width

Hidden state

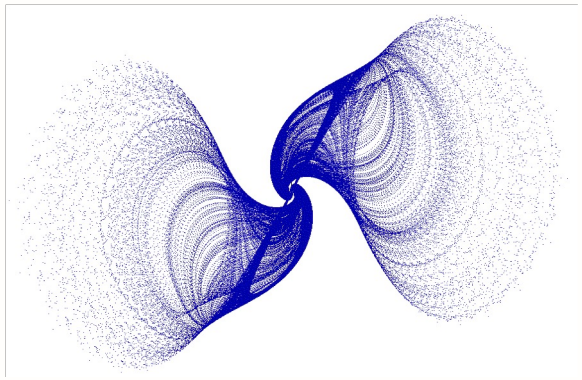
Inputs

Model parameters

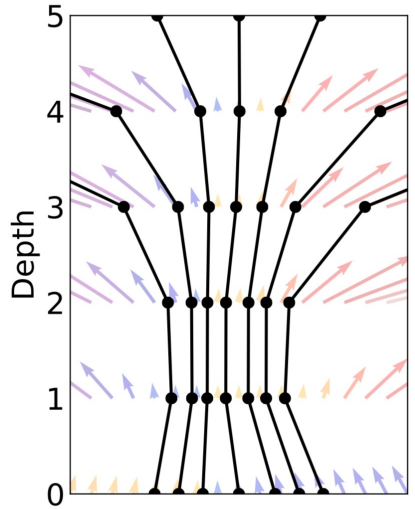
activations



Dynamical systems



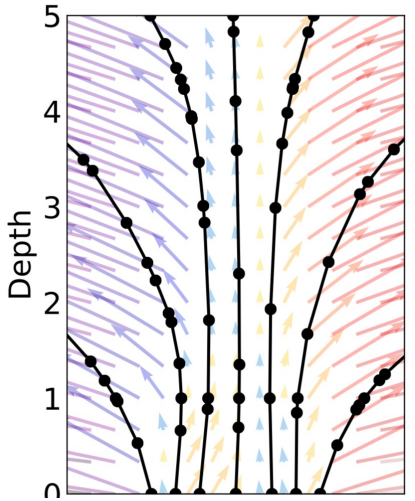
Residual Network



$$\mathbf{h}_{t+1} = \mathbf{h}_t + f(\mathbf{h}_t, \theta_t)$$

He et al.
CVPR 2016

ODE Network



$$\frac{d\mathbf{h}(t)}{dt} = f(\mathbf{h}(t), t, \theta)$$

Chen et al.
NeurIPS 2018

Figure Credit: Chen et al. NeurIPS 2018

What is a continuous- time/depth neural network?

Standard Recurrent Neural Network (RNN)
Hopfield 1982

$$x(t+1) = f(x(t), I(t), t\,; \theta)$$

Neural ODE
Chen et al. NeurIPS, 2018

$$\frac{dx(t)}{dt} = f(x(t), I(t), t\,; \theta)$$

Continuous-time (CT) RNN
Funahashi et al. 1993

$$\frac{dx(t)}{dt} = -\frac{x(t)}{\tau} + f(x(t), I(t), t\,; \theta)$$

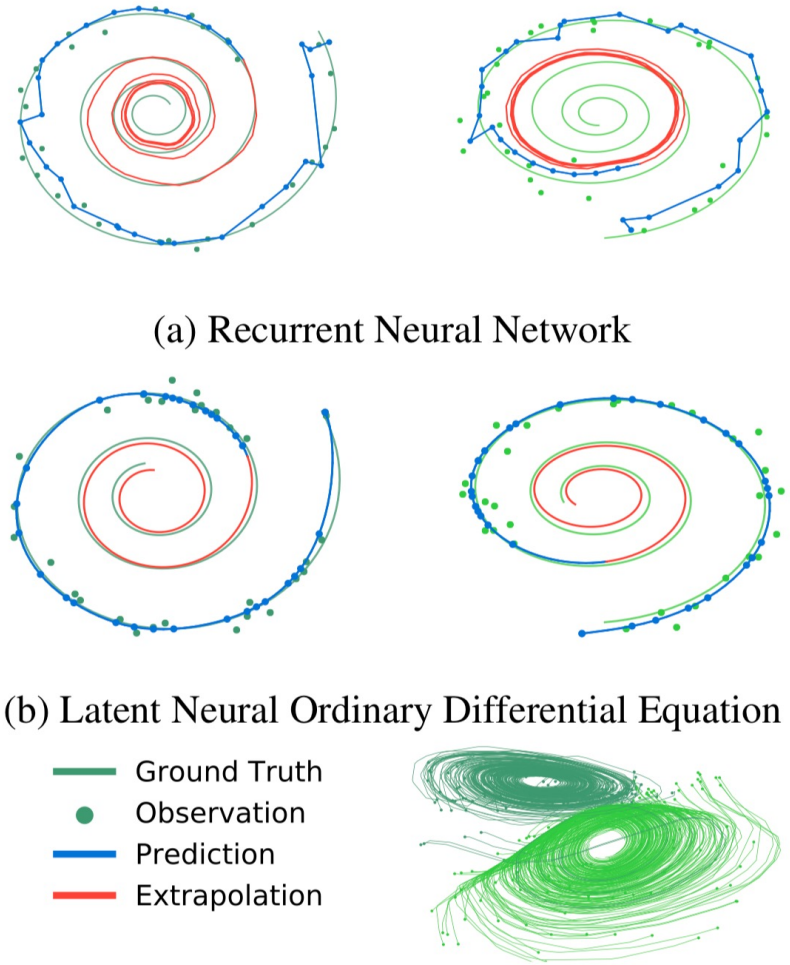


Figure Credit: Chen et al. NeurIPS 2018

Liquid Time-Constant (LTC) networks

1. Linear state-space model

$$d\mathbf{x}(t)/dt = -\mathbf{x}(t)/\tau + \mathbf{S}(t) \qquad \mathbf{S}(t) \in \mathbb{R}^M$$

2. Non-linear synapse Model

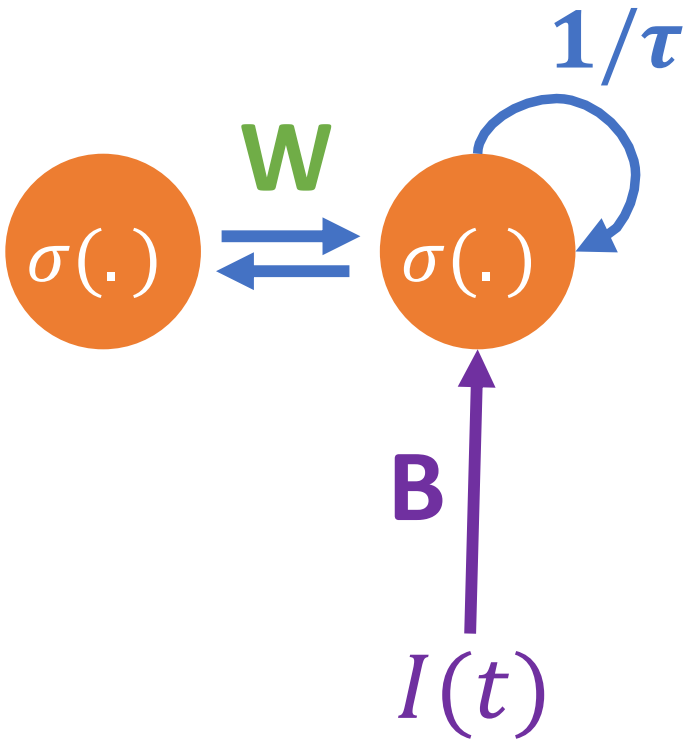
$$\mathbf{S}(t) = f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)(A - \mathbf{x}(t))$$

$$\frac{d\mathbf{x}(t)}{dt} = -\left[\frac{1}{\tau} + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right]\mathbf{x}(t) + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)A$$

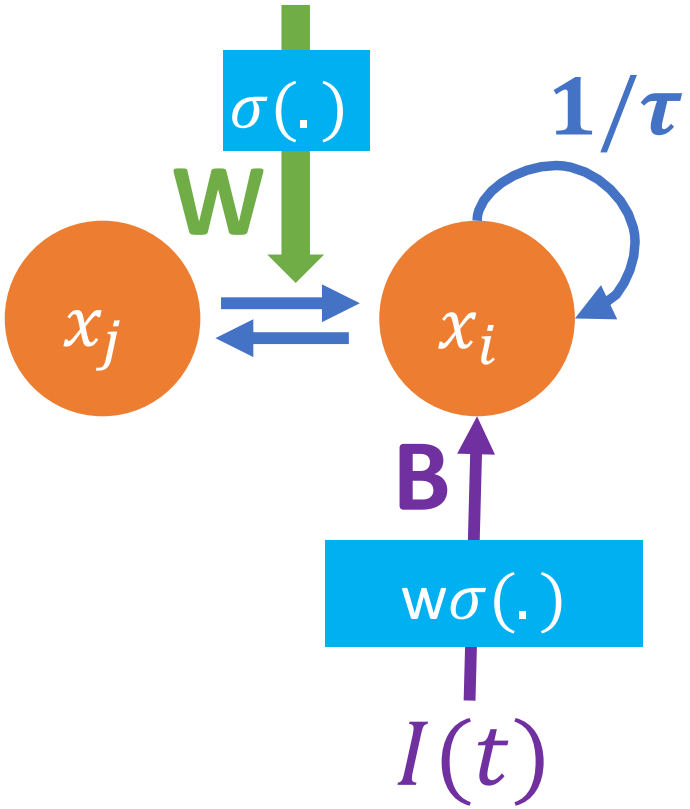
"Liquid" = variable

Liquid Time-Constant (LTC) networks

Standard Neural Nets



Liquid Neural Nets

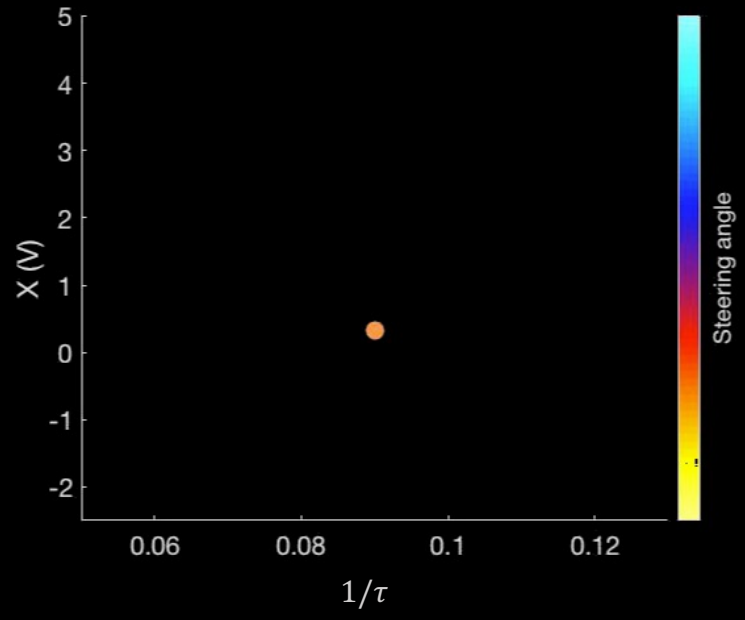


$1/\tau$ intrinsic coupling

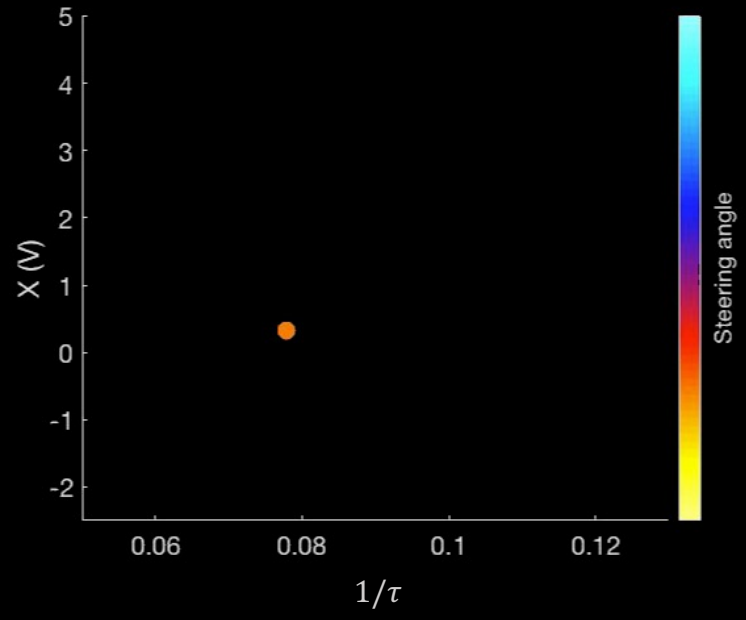
$\mathbf{W}\sigma(.)$ liquidity modulator

B input regulator

Standard Neural Network



Liquid Neural Network



LTCs have stable state and time-constant

$$\frac{d\mathbf{x}(t)}{dt} = -\left[\frac{1}{\tau} + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right]\mathbf{x}(t) + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)A$$

System time-constant

$$\tau_{sys} = \frac{\tau}{1+\tau f(\mathbf{x}(t),\mathbf{I}(t),t,\theta)}$$

Theorem 1. Time-constant of each LTC neuron is bounded:

$$\tau_i/(1+\tau_i W_i) \leq \tau_{sys_i} \leq \tau_i$$

Theorem 2. The state of each LTC neuron is bounded

$$min(0, A_i^{min}) \leq x_i(t) \leq max(0, A_i^{max})$$

Liquid Time-Constant Networks are Universal Approximators

Theorem 3. LTCs are universal approximators

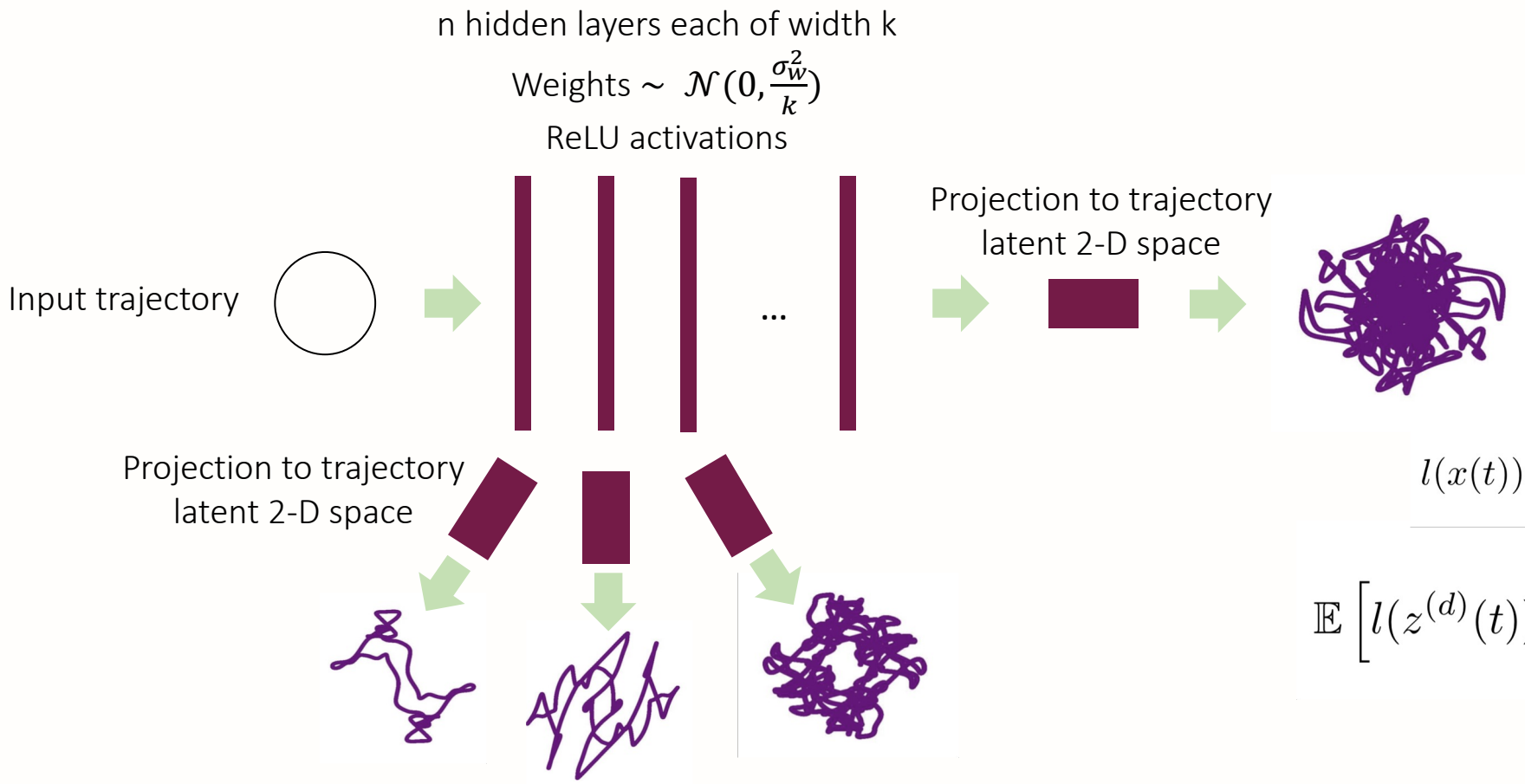
$$\frac{d\mathbf{x}(t)}{dt} = -\left[\frac{1}{\tau} + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right]\mathbf{x}(t) + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)A$$

Expressivity

Defining a better measure

Trajectory Length as a measure of expressivity of Deep networks

[Raghu et al. ICML 2017]



$$l(x(t)) = \int_t \left|\left| \frac{dx(t)}{dt} \right|\right| dt$$ Arc-length

$$\mathbb{E}\left[l(z^{(d)}(t)) \right] \geq O\left(\frac{\sigma_w \sqrt{k}}{\sqrt{k+1}} \right)^d l(x(t))$$

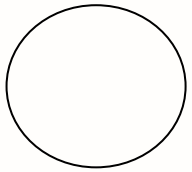
[Raghu et al. ICML 2017]

Expressivity

Trajectory length as a measure of expressivity

Let's implement the trajectory space for time-continuous models

Input trajectory



LTC
CT-RNN
Neural ODE

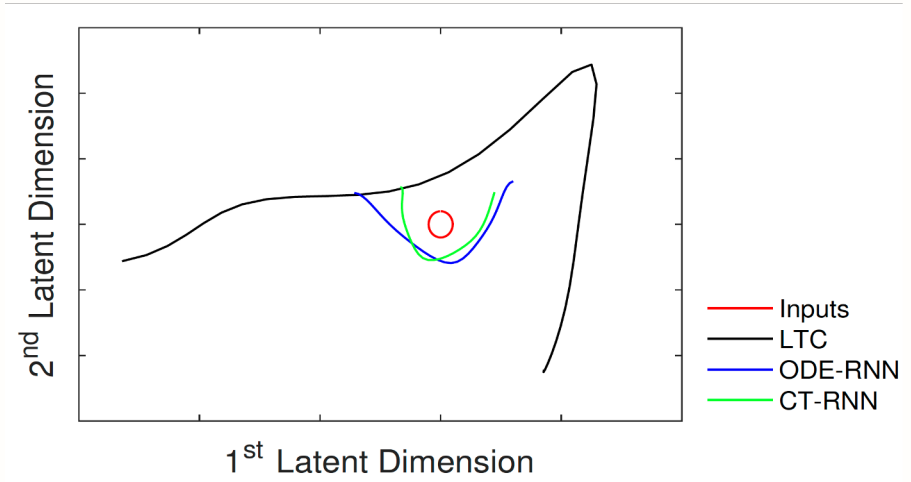
Weights ~ $\mathcal{N}(0, \frac{\sigma_w^2}{k})$

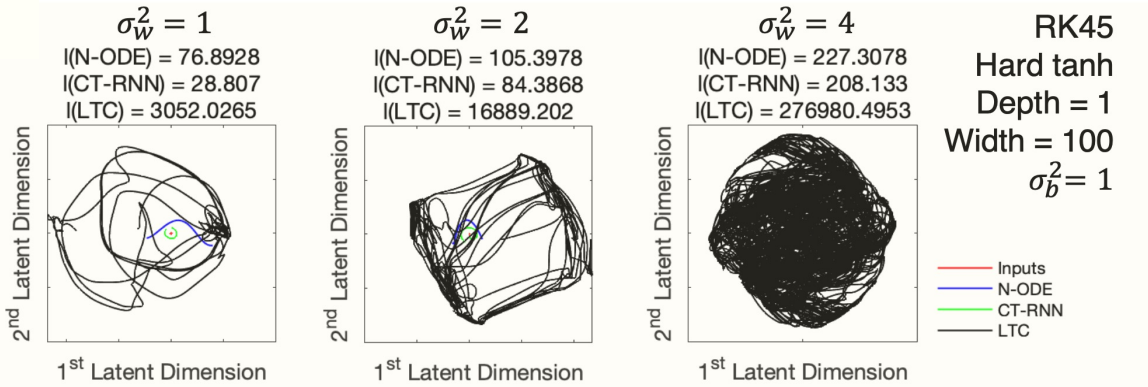
activation functions:
ReLU, tanh, logistic sigmoid

Projection to a latent trajectory 2-D space

PCA

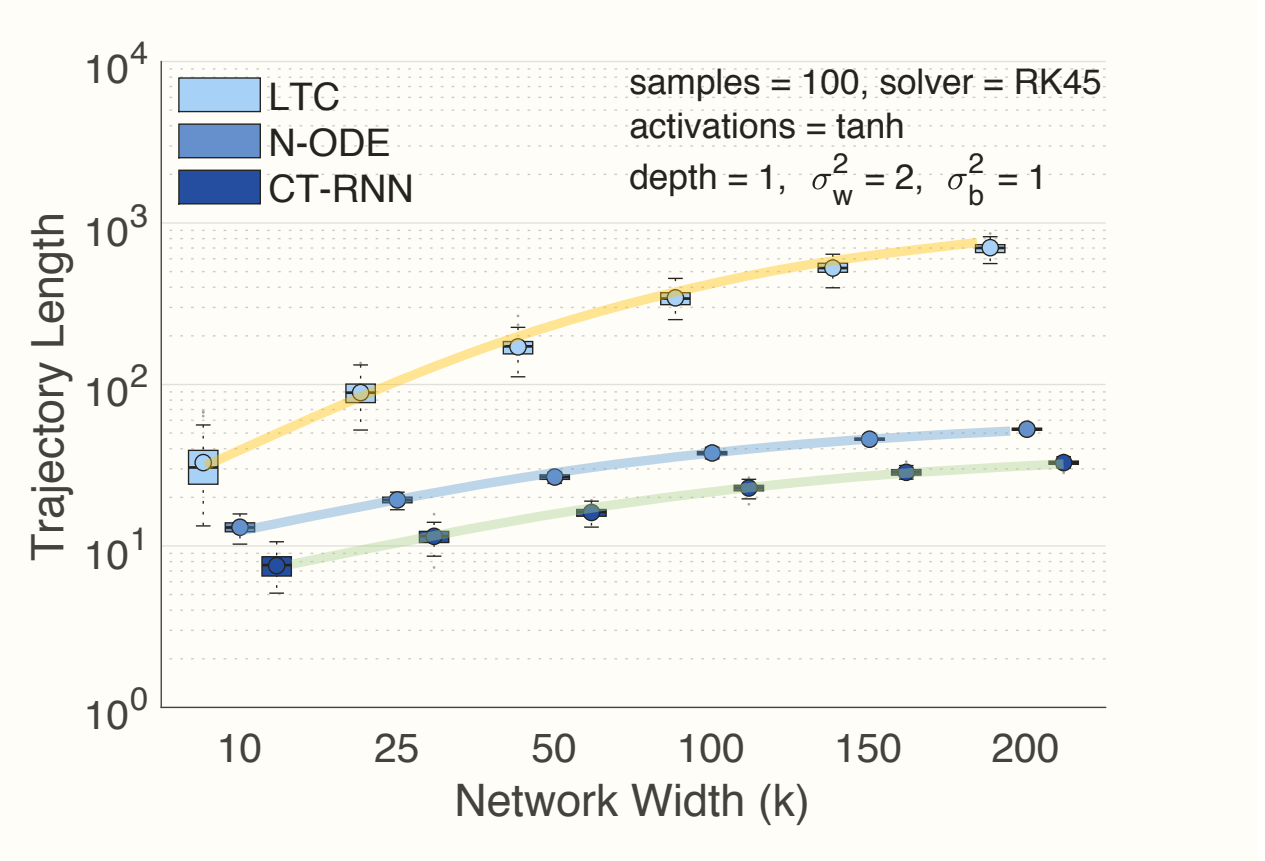
PCA = Principle Component Analysis





Expressivity

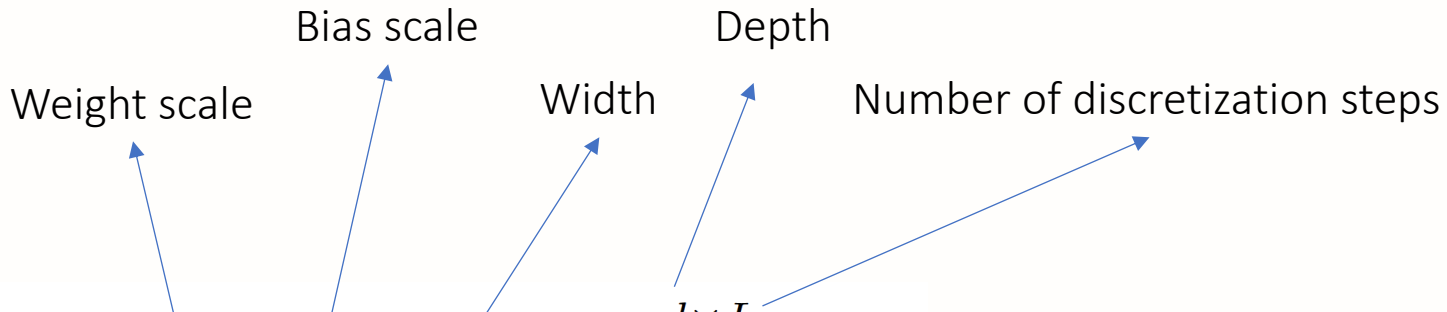
Trajectory length as a measure of expressivity



Hasani et al. AAAI 2021

Expressivity

Trajectory length lower bound



$$\text{Neural ODE:}\quad \mathbb{E}\left[l(z^{(d)}(t))\right] \geq O\left(\frac{\sigma_w\sqrt{k}}{\sqrt{\sigma_w^2+\sigma_b^2+k\sqrt{\sigma_w^2+\sigma_b^2}}}\right)^{d\times L} l(I(t))$$

$$\text{CT-RNN:}\quad \mathbb{E}\left[l(z^{(d)}(t))\right] \geq O\left(\frac{(\sigma_w-\sigma_b)\sqrt{k}}{\sqrt{\sigma_w^2+\sigma_b^2+k\sqrt{\sigma_w^2+\sigma_b^2}}}\right)^{d\times L} l(I(t))$$

$$\text{LTC:}\quad \mathbb{E}\left[l(z^{(d)}(t))\right] \geq O\left(\left(\frac{\sigma_w\sqrt{k}}{\sqrt{\sigma_w^2+\sigma_b^2+k\sqrt{\sigma_w^2+\sigma_b^2}}}\right)^{d\times L}\left(\sigma_w+\frac{\|z^{(d)}\|}{\min(\delta t,L)}\right)\right) l(I(t))$$

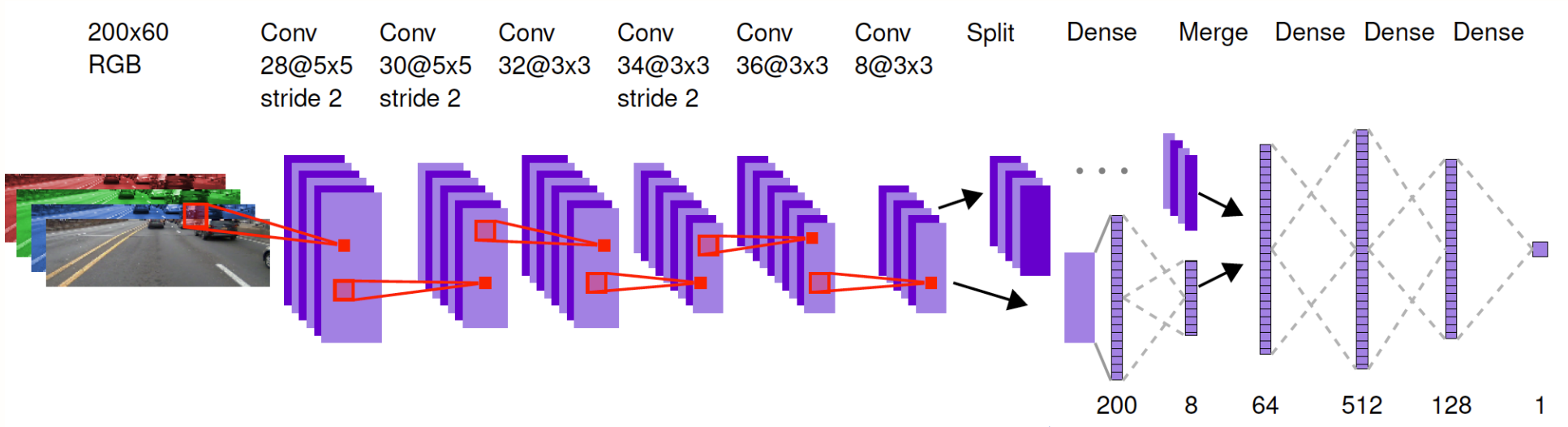
System's dynamic time-scale

Outline

- ✓ Liquid time-constant networks
- ✓ **Autonomy with liquid networks**
- ✓ Causality
- ✓ Closed-form Continuous-time decision-making
- ✓ Task understanding and out of distribution generalization

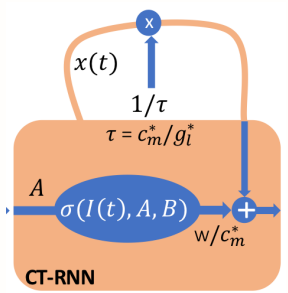
LTCs: Performance

High-fidelity autonomy by LTCs - end-to-end learning

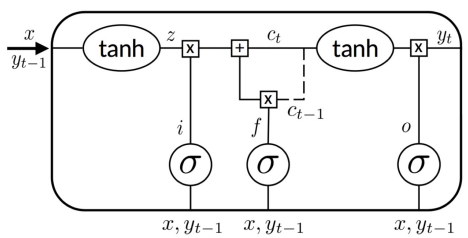


What if we replace the fully connected layers by a recurrent neural network?

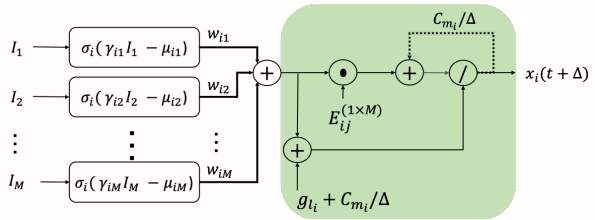
ODE-RNN



LSTMs



LTC-based Networks?



Lechner*, Hasani* et al. Nature MI, 2020

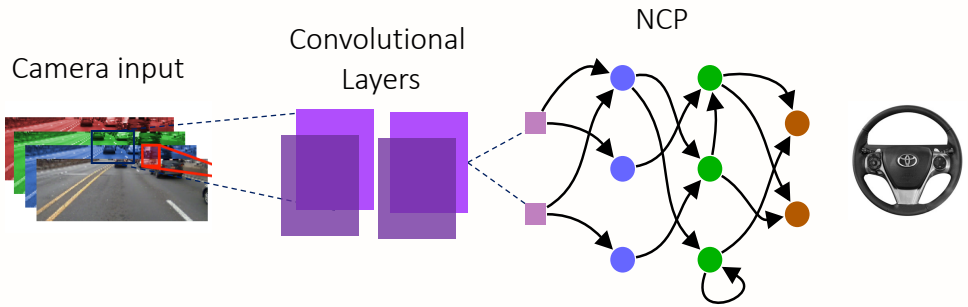
LTCs: Performance

High-fidelity autonomy by LTCs

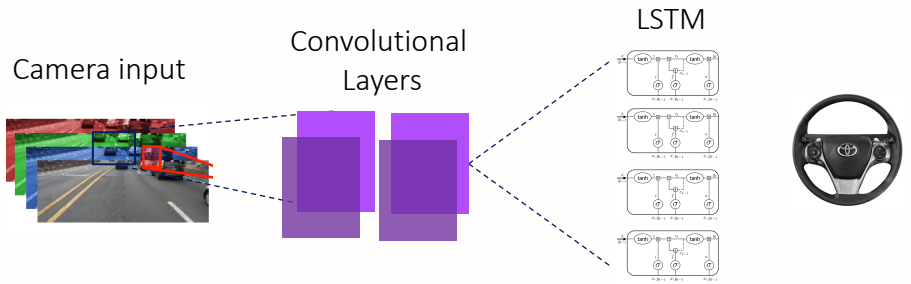
end-to-end learning of Neural Circuit Policies (NCP)

Now we compare properties of NCPs with a number of other models

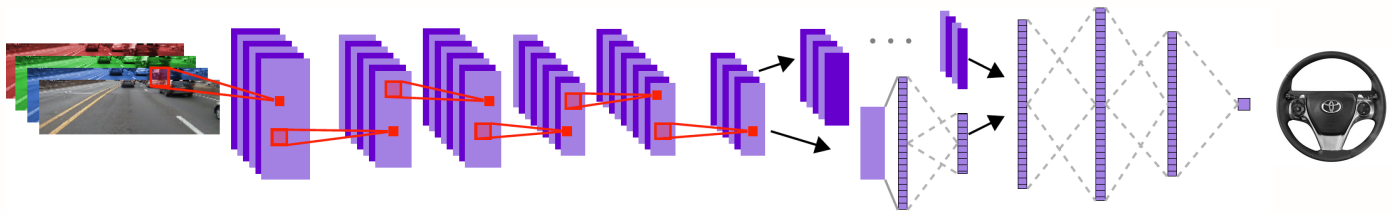
NCPs



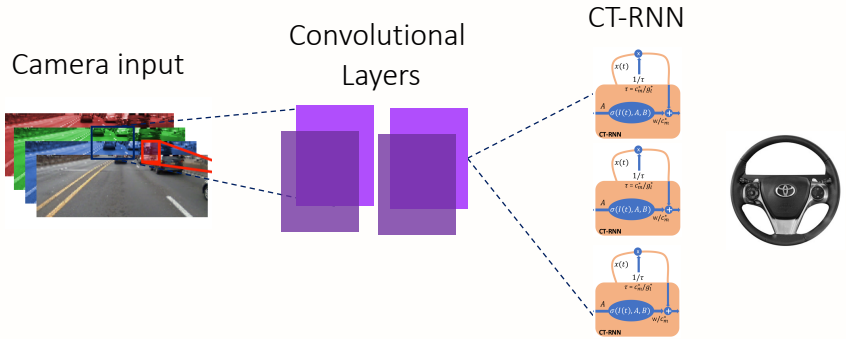
LSTMs



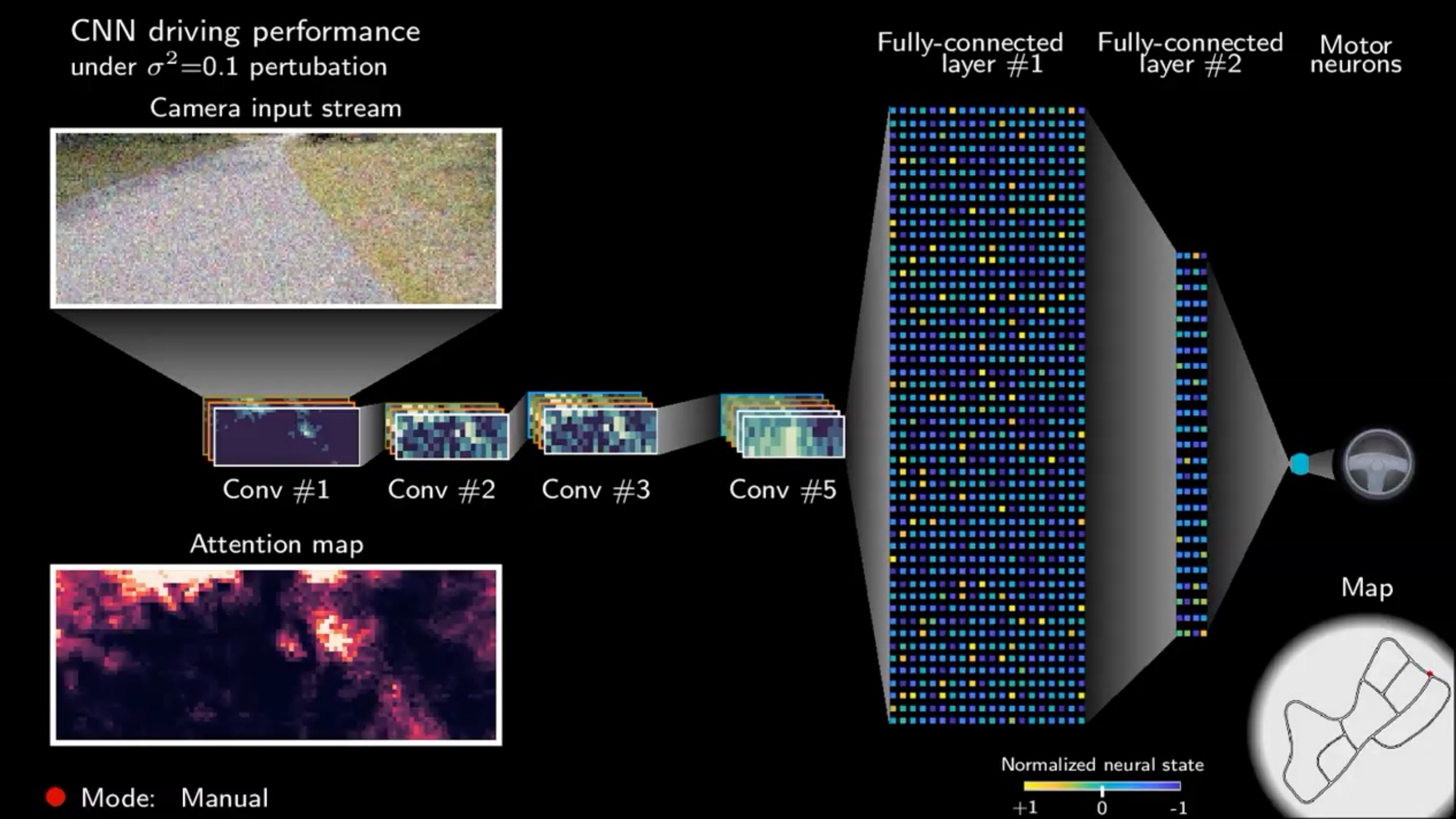
Convolutional Neural Networks (CNNs)

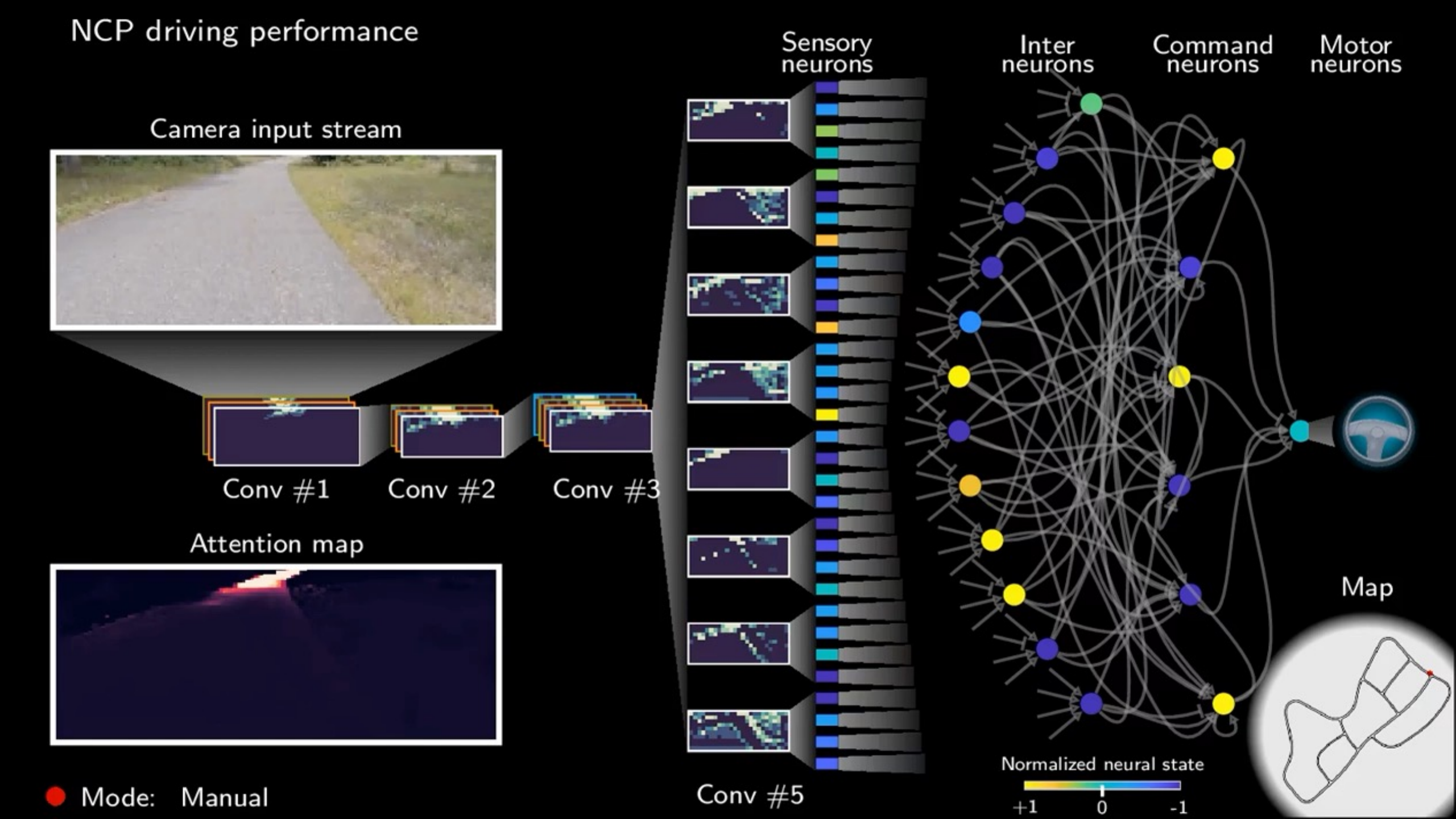


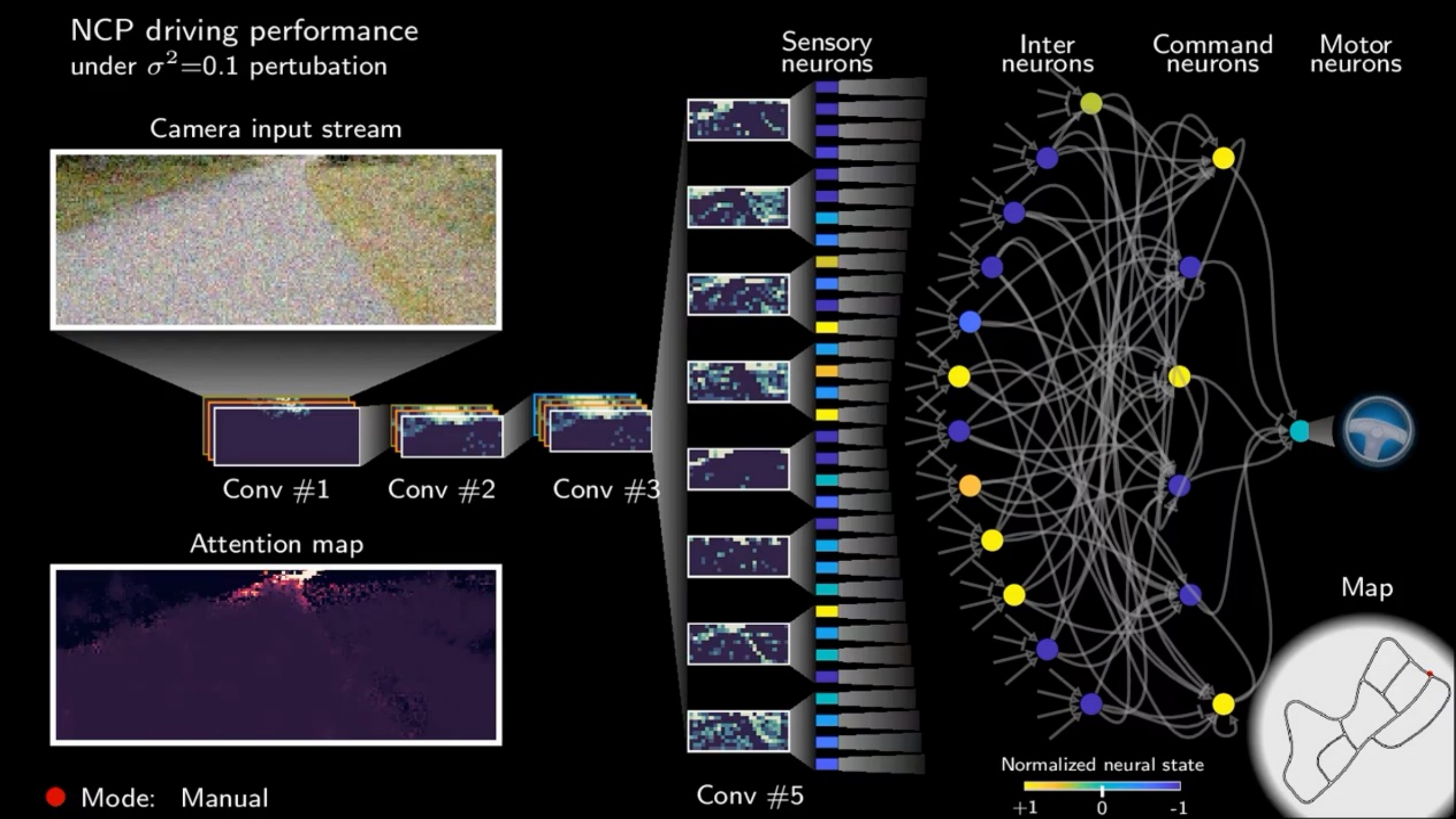
CT-RNNs



Lechner*, Hasani* et al. Nature MI, 2020


CNN driving performance
under σ^2=0.1 pertubation
Camera input stream
Attention map
Conv #1
Conv #2
Conv #3
Conv #5
Fully-connected layer #1
Fully-connected layer #2
Motor neurons
Map
Normalized neural state
+1
0
-1
Mode: Manual


NCP driving performance
Camera input stream
Conv #1
Conv #2
Conv #3
Attention map
Sensory neurons
Conv #5
Inter neurons
Command neurons
Motor neurons
Map
Normalized neural state
+1
0
-1
Mode: Manual

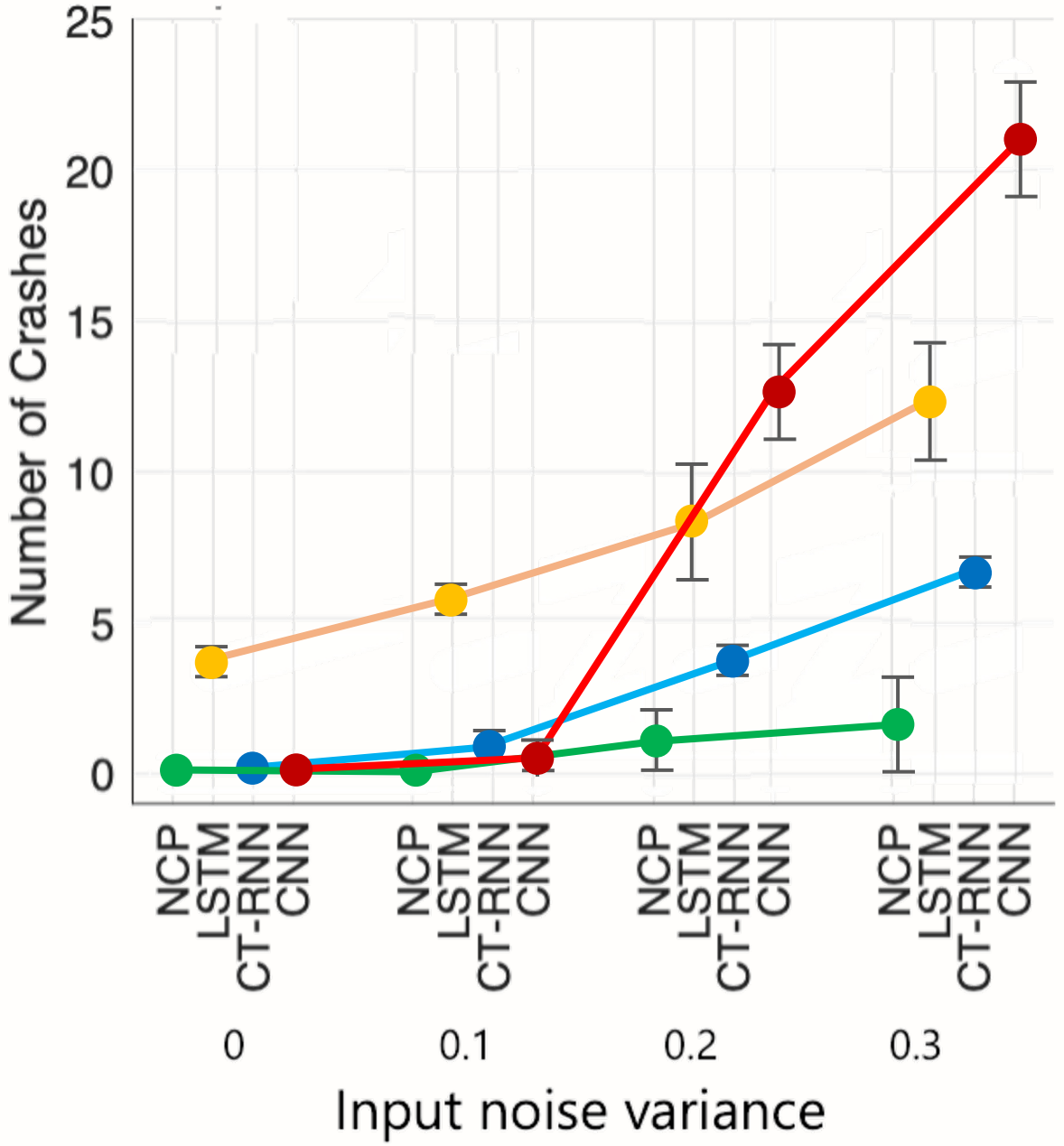

NCP driving performance
under σ^2=0.1 pertubation
Camera input stream
Conv #1
Conv #2
Conv #3
Attention map
Sensory neurons
Conv #5
Inter neurons
Command neurons
Motor neurons
Normalized neural state
+1
0
-1
Map
Mode: Manual

LTCs: Performance

Robustness to perturbations

CNN
LSTM
CT-RNN
LTC (ours)

Liquid time-constant neuron resilience to sensory noise

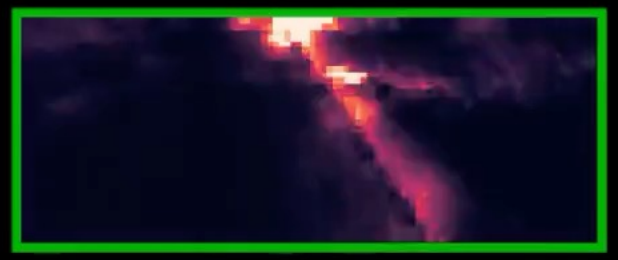


Lechner*, Hasani* et al. Nature MI, 2020

Camera input stream

Map

CNN



Mode: Autonomous

CT-RNN

Mode: Autonomous

LSTM

Mode: Autonomous

Our solution

Mode: Autonomous

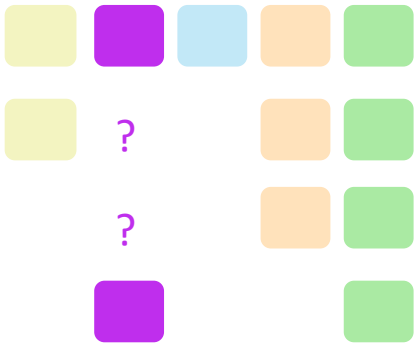
Outline

- ✓ Liquid time-constant networks
- ✓ Autonomy with liquid networks
- ✓ **Causality**
- ✓ Closed-form Continuous-time decision-making
- ✓ Task understanding and out of distribution generalization

Why can LTCs learn better causal relationships?

Taxonomy of Models

Adapted from: Peters, Janzing, Schölkopf, MIT Press, 2017



Physical

Structural causal

Causal graphs

Statistical models

Insights about system

Learn from data

Answer counter factual

Predict in IID

Interventions

Dynamic Causal Models

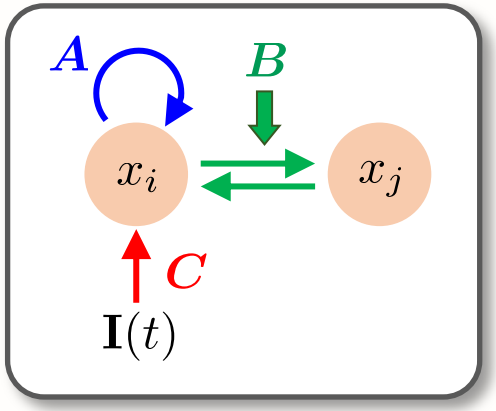
$$\frac{d\mathbf{x}}{dt} = g(\mathbf{x}(t), \mathbf{I}(t); \theta)$$

$$(A + \mathbf{I}(t)B)\,\mathbf{x}(t) + C(\mathbf{I}(t))$$

Internal coupling

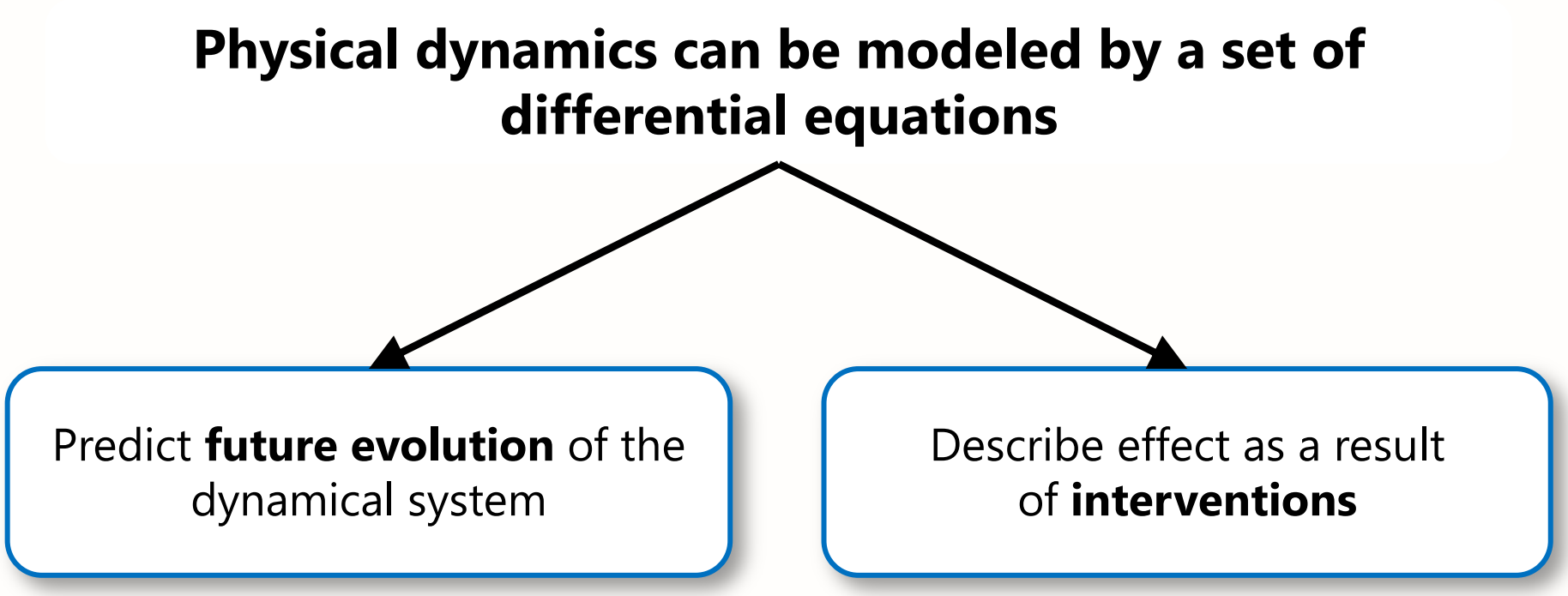
Internal Intervention

External Intervention



The **LTC** model reduces mathematically to a **Dynamic Causal Model**

Differential equations can form causal structures



(Friston et al., 2003)

Differential equations can form causal structures

Given the following system of differential equations:

$$\frac{d\mathbf{x}}{dt} = g(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^d, \quad \mathbf{x}(0) = x_0, \quad g(\mathbf{x}) = \text{nonlinearity}$

Picard-Lindelöf theorem

(Nevanlinna, 1989)

states that above DE has a unique solution as long as g is Lipschitz

Euler solution

The Euler method unrolling:

$$\mathbf{x}(t + \delta t) = \mathbf{x}(t) + dt\, g(\mathbf{x})$$

Causal structure

(Schölkopf, 2019)

Representation under uniqueness conditions forms a temporally **causal structure**

Dynamic Causal Models (DCMs)

$$\frac{d\mathbf{x}}{dt} = g(\mathbf{x}(t), \mathbf{I}(t); \theta)$$

Bilinear approximation

(Friston et al., 2003)

$$\frac{d\mathbf{x}}{dt} = (A + \mathbf{I}(t)B)\,\mathbf{x}(t) + C(\mathbf{I}(t))$$

Internal coupling

$$A = \left.\frac{\partial g}{\partial \mathbf{x}}\right|_{\mathbf{I}=0}$$

Regulates hidden state

Internal Intervention

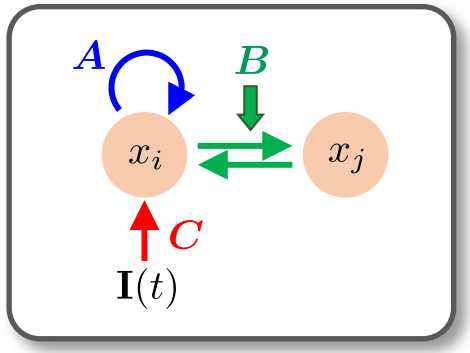
$$B = \frac{\partial^2 g}{\partial \mathbf{x}\,\partial \mathbf{I}}$$

Controls coupling sensitivity among network's nodes

External Intervention

$$C = \left.\frac{\partial g}{\partial \mathbf{I}}\right|_{\mathbf{x}=0}$$

Regulates external input



The **Liquid Time-constant (LTC)** model reduces to a **Dynamic Causal Model** of this form if

1. $g(\cdot)$ **is continuous and bounded**

$e.g.,\quad \tanh(W_r\mathbf{x}(t) + W\mathbf{I}(t) + b)$

(Hirsch and Smale, 1973, Hasani, et al. 2021)

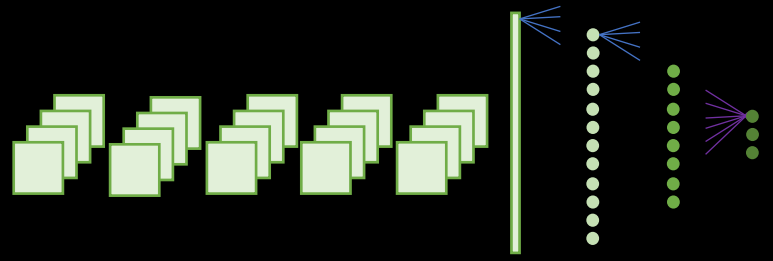
2. τ **is positive**

Enforced by activation constraint

(Funahashi and Nakamura, 1993)

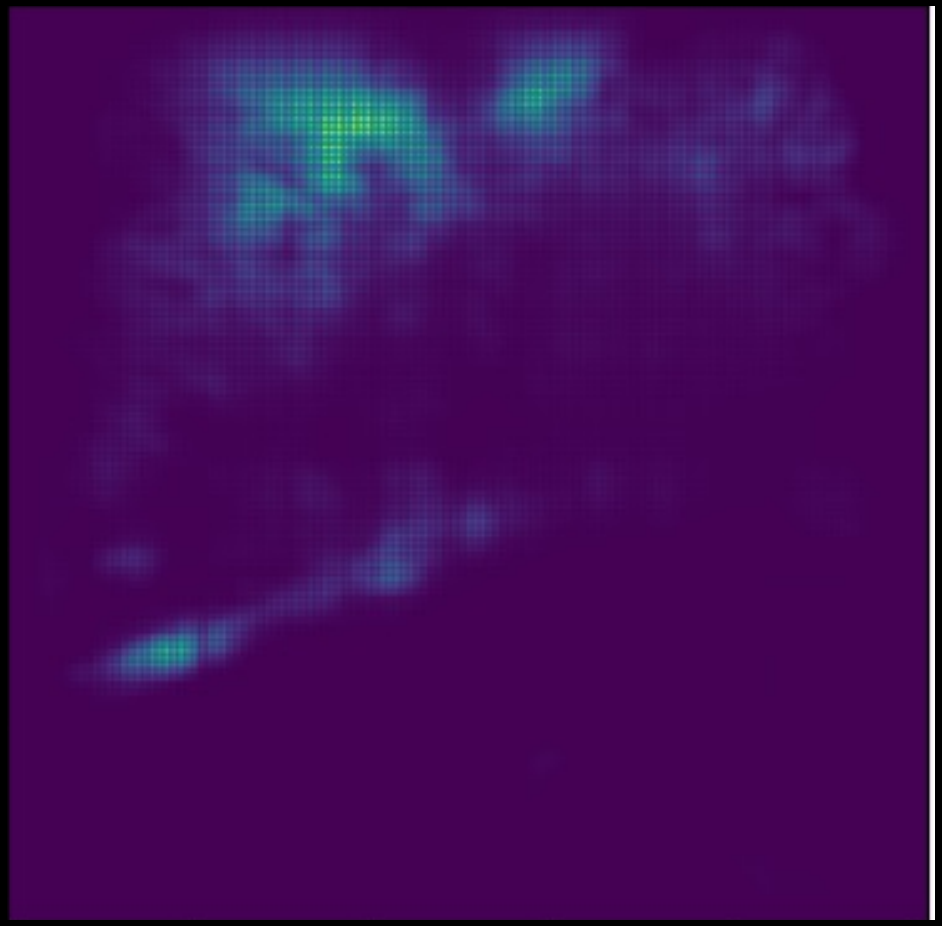
LTC-based: Neural Circuit Policies

Performance – Attention



Flying Performance

Visual Backprop Attention Map

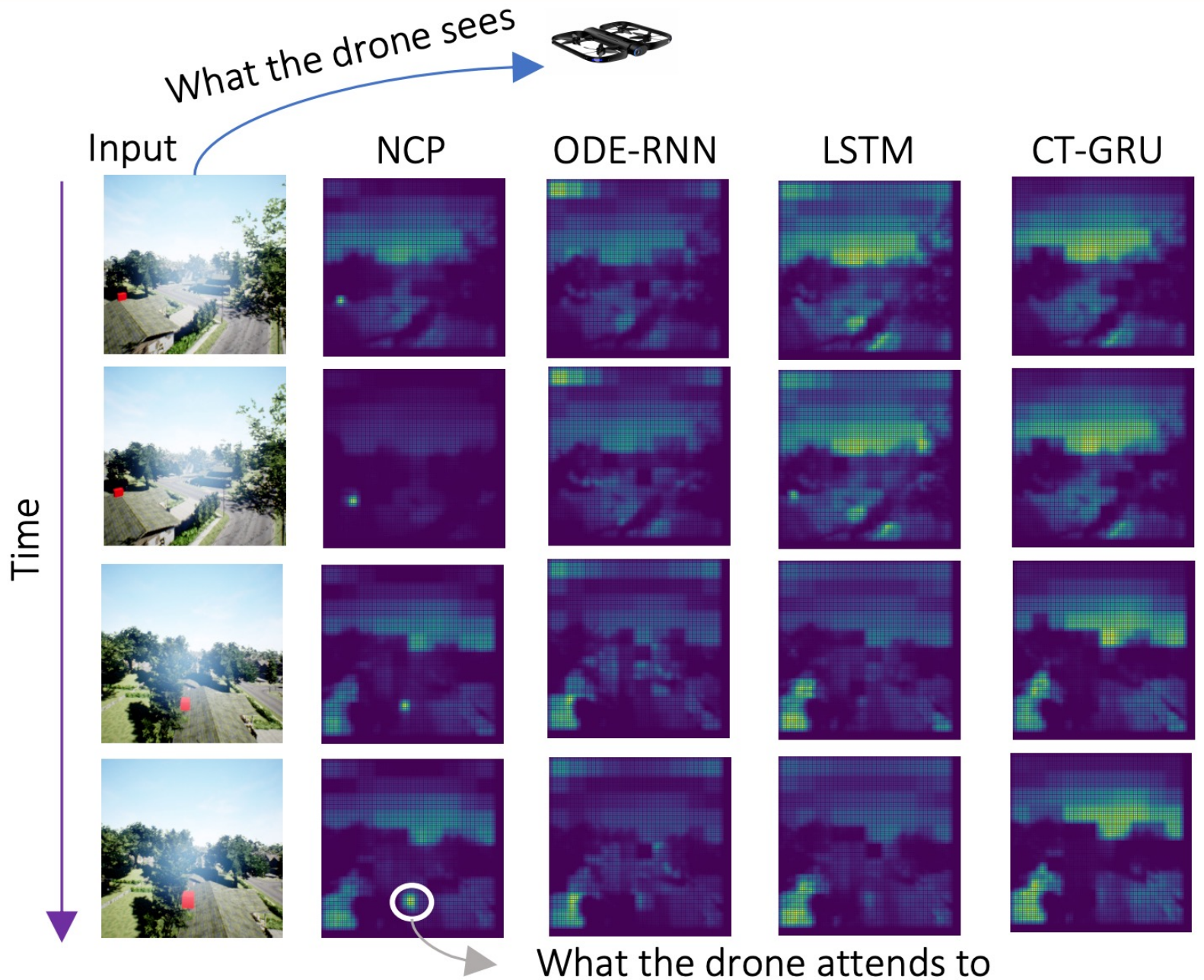


Vorbach*, Hasani*, et al. NeurIPS 2021

Neural Circuit Policies

Performance – Attention

- Red cubic target is fixed.
- Drone learns to navigate to target by visual inputs only



Liquid Network's Limitations

- The complexity of LTC-based networks depends on the complexity of their ODE-solvers
  → long training and test time for larger networks
    solution: 1) use stable fixed-step ODE-solvers [Hasani et al, AAAI 2021]
    2) use Sparse Flows [Liebenwein* & Hasani * et al, NeurIPS 2021]
    3) use Hypersolvers [Poli et al, NeurIPS 2020]
    4) **use Closed-form variants** [Hasani et al, Nature MI, 2022]

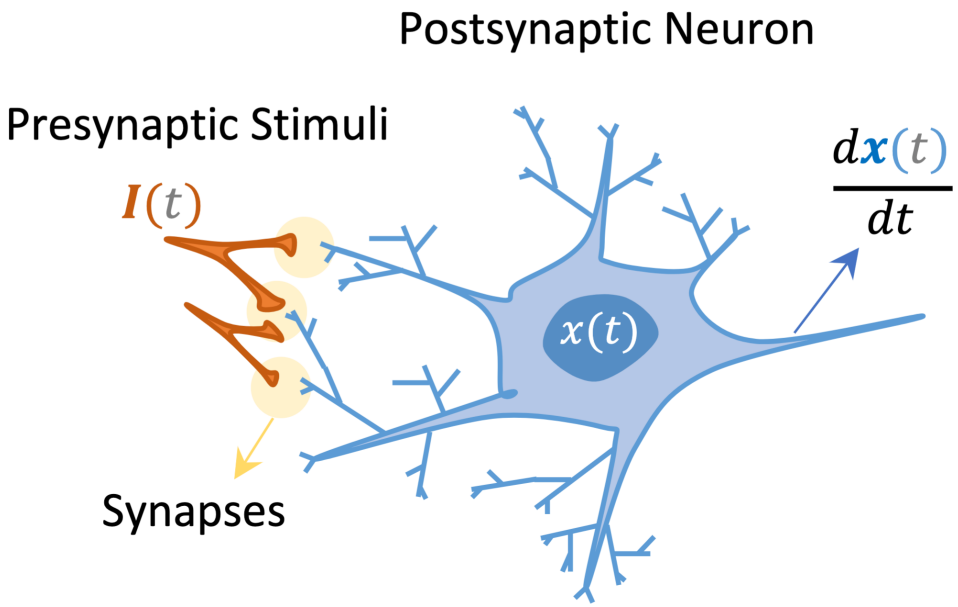
- Similar to all ODE-based networks they might express vanishing/exploding gradients issues
  → Face difficulties in learning very long sequences
    solution: 1) use mixed memory wrappers with them: [Lechner and Hasani, MemARI NeurIPS 2022]
    2) Liquid Structural State-Space Models [Hasani et al. ICLR 2023]

Outline

- ✓ Liquid time-constant networks
- ✓ Autonomy with liquid networks
- ✓ Causality
- ✓ **Closed-form Continuous-time decision-making**
- ✓ Task understanding and out of distribution generalization

Closed-form Solution of Liquid Networks

Closed-form Continuous-time Neural Networks



$$S(t) = f\big(\boldsymbol{I}(t)\big)\,(A - \boldsymbol{x}(t))$$

$$\frac{d\boldsymbol{x}(t)}{dt} = -\frac{\boldsymbol{x}(t)}{\tau} + S(t)$$

this is a liquid time-constant differential equation instance

we solve this in closed-form

$$\boldsymbol{x}(t) = (\boldsymbol{x}(0) - A)\, e^{-\left[\frac{1}{\tau} + f(I(t))\right]t}\, f(-I(t)) + A$$

$\boldsymbol{x}(t)$	Postsynaptic neuron's potential
A	Synaptic reversal potential
$f(.)$	Synaptic release nonlinearity
τ	Postsynaptic neuron's time-constant

Closed-form Solution of Liquid Networks

Closed-form Continuous-time Neural Networks (CfCs)

Time Complexity of the process to compute K solver's steps

Method	Complexity	Local Error
p-th order solver	$\mathcal{O}(K \cdot p)$	$\mathcal{O}(\epsilon^{p+1})$
adaptive–step solver	–	$\mathcal{O}(\tilde{\epsilon}^{\,p+1})$
Euler `hypersolver`	$\mathcal{O}(K)$	$\mathcal{O}(\delta\epsilon^2)$
p-th order `hypersolver`	$\mathcal{O}(K \cdot p)$	$\mathcal{O}(\delta\epsilon^{p+1})$
CfC (Ours)	$\mathcal{O}(\tilde{K})$	**not relevant**

ϵ is step size

$\tilde{\epsilon}$ is the maximum step size

$\delta \ll 0$

$\tilde{K}$ is the input time steps

Hasani et al. Nature MI 2022

Closed-form Solution of Liquid Networks

Closed-form Continuous-time Neural Networks (CfCs)

Sequence and time-step prediction complexity

Model	Sequence prediction	Time-step prediction
RNN	$\mathcal{O}(nk)$	$\mathcal{O}(k)$
ODE-RNN	$\mathcal{O}(nkp)$	$\mathcal{O}(kp)$
Transformer	$\mathcal{O}(n^2k)$	$\mathcal{O}(nk)$
CfC	$\mathcal{O}(nk)$	$\mathcal{O}(k)$

n is sequence length
k is the number of hidden units
p is the order of the ODE-solver

Hasani et al. Nature MI 2022

Deriving the Closed-form Solution of LTCs

Theorem

For a given LTC initial-value problem determined by $\frac{d\mathbf{x}}{dt} = -(w_\tau + f(\mathbf{x}, \mathbf{I}, \theta))\mathbf{x}(t) + Af(\mathbf{x}, \mathbf{I}, \theta)$,
Constructed by one cell,
receiving a single dimensional time-series input $I(t)$
with no self-connections,
the following expression $\tilde{x}(t) = (x(0) - A)e^{-\left[w_\tau t + f(I(t))t\right]} f(-I(t)) + A$
is an approximation of its closed-form solution by the following sharpness results:

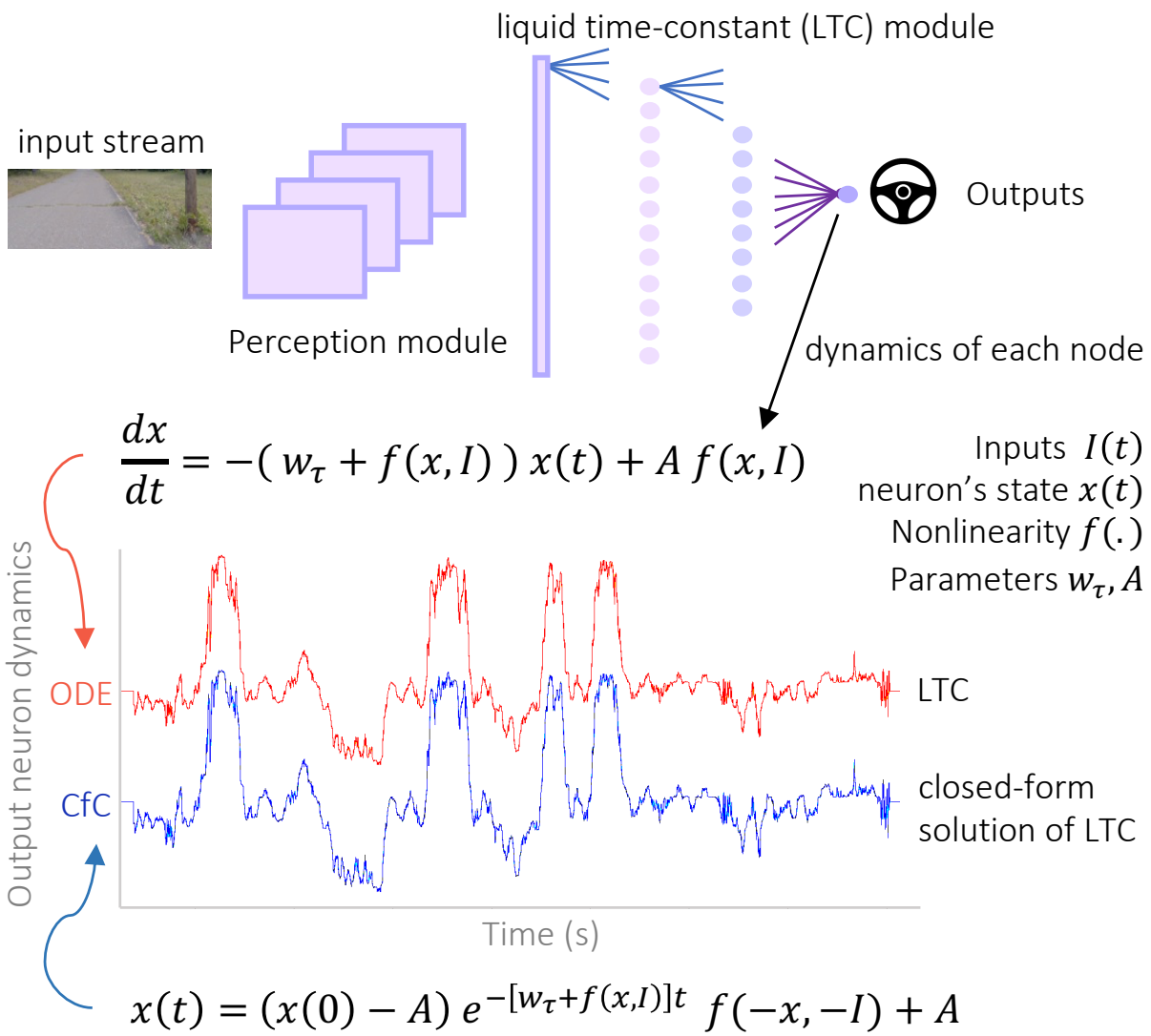
$$|x(t) - \tilde{x}(t)| \leq |x(0) - A|e^{-w_\tau t} \text{ for all } t \geq 0$$

1. For any $t \geq 0$, we have $\sup\{\frac{1}{c}(x(t) - \tilde{x}(t)) \mid I : [0; t] \to \mathbb{R}\} = e^{-w_\tau t}$.

2. For any $t \geq 0$, we have $\inf\{\frac{1}{c}(x(t) - \tilde{x}(t)) \mid I : [0; t] \to \mathbb{R}\} = e^{-w_\tau t}(e^{-t} - 1)$.

$c = x(0) - A$

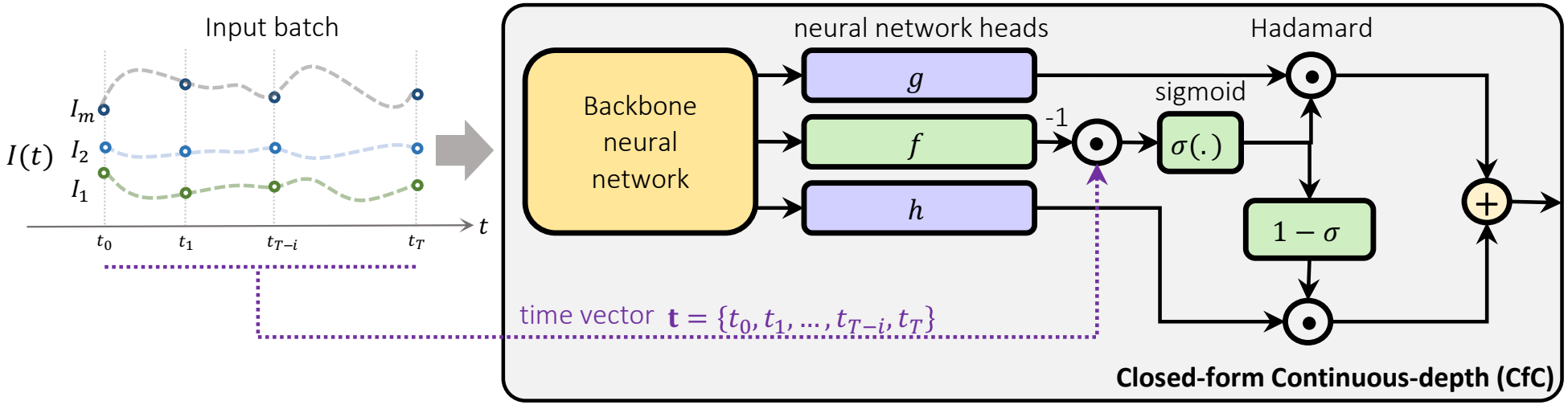
Tightness of the Closed-form Solution in Practice



Hasani et al. Nature MI 2022

Closed-form Continuous-time Neural Networks

Liquid CfCs



$$\mathbf{x}(t) = \underbrace{\sigma(-f(\mathbf{x}, \mathbf{I}; \theta_f)\ \mathbf{t})}_{\text{time-continuous gating}} \odot g(\mathbf{x}, \mathbf{I}; \theta_g) + \underbrace{\left[1 - \sigma(-[f(\mathbf{x}, \mathbf{I}; \theta_f)]\ \mathbf{t})\right]}_{\text{time-continuous gating}} \odot h(\mathbf{x}, \mathbf{I}; \theta_h)$$

LTC's Closed-form solution

$$\tilde{x}(t) = (x(0) - A)e^{-\left[w_\tau t + f(I(t))t\right]} f(-I(t)) + A$$

Liquid ODE

$$\frac{d\mathbf{x}}{dt} = -(w_\tau + f(\mathbf{x}, \mathbf{I}, \theta))\mathbf{x}(t) + Af(\mathbf{x}, \mathbf{I}, \theta)$$

How Well liquid CfCs perform in Time-series modeling?

Physical Dynamics Modeling

Table 6: **Per time-step regression**. Walker2d kinematic dataset. (mean $\pm$ std, $N = 5$)

Model	Square-error
†ODE-RNN (Rubanova et al., 2019)	1.904 $\pm$ 0.061
†CT-RNN (Funahashi and Nakamura, 1993)	1.198 $\pm$ 0.004
†Augmented LSTM (Hochreiter and Schmidhuber, 1997)	1.065 $\pm$ 0.006
†CT-GRU (Mozer et al., 2017)	1.172 $\pm$ 0.011
†RNN-Decay (Rubanova et al., 2019)	1.406 $\pm$ 0.005
†Bi-directional RNN (Schuster and Paliwal, 1997)	1.071 $\pm$ 0.009
†GRU-D (Che et al., 2018)	1.090 $\pm$ 0.034
†PhasedLSTM (Neil et al., 2016)	1.063 $\pm$ 0.010
†GRU-ODE (Rubanova et al., 2019)	1.051 $\pm$ 0.018
†CT-LSTM (Mei and Eisner, 2017)	1.014 $\pm$ 0.014
†ODE-LSTM (Lechner and Hasani, 2020)	0.883 $\pm$ 0.014
coRNN (Rusch and Mishra, 2021)	3.241 $\pm$ 0.215
Lipschitz RNN (Erichson et al., 2021)	1.781 $\pm$ 0.013
LTC (Hasani et al., 2021)	**0.662** $\pm$ 0.013
Transformer (Vaswani et al., 2017)	0.761 $\pm$ 0.032
Cf-S (ours)	0.948 $\pm$ 0.009
CfC-noGate (ours)	**0.650** $\pm$ 0.008
CfC (ours)	**0.643** $\pm$ 0.006
CfC-mmRNN (ours)	**0.617** $\pm$ 0.006

Outline

- ✓ Liquid time-constant networks
- ✓ Autonomy with liquid networks
- ✓ Causality
- ✓ Closed-form Continuous-time decision-making
- ✓ **Task understanding and out of distribution generalization**

Can AI Systems Understand the Task They Are Given?

Fly-to-target tasks

Chahine* & Hasani* et al. Science Robotics 2023

Liquid Networks in closed-loop Aerial Vehicle Control

Sample Training Data

Chahine* & Hasani* et al. Science Robotics 2023

Standard Deep Neural Network

Task: Identify and navigate to target
Deploy: Closed-loop testing
Result: Spurious attention → Fail

Agent

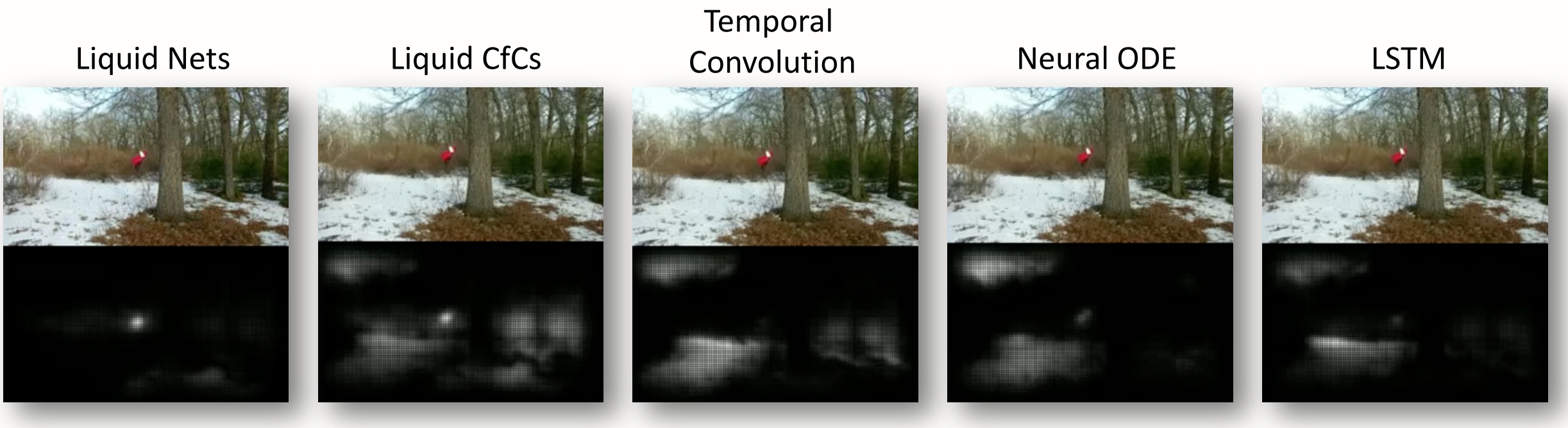
Target

LTC-based Network

Task: Identify and navigate to target
Deploy: Closed-loop testing
Result: Causal attention → Success

Generalization and Task understanding with Liquid Networks

Testing attention profiles of different networks offline

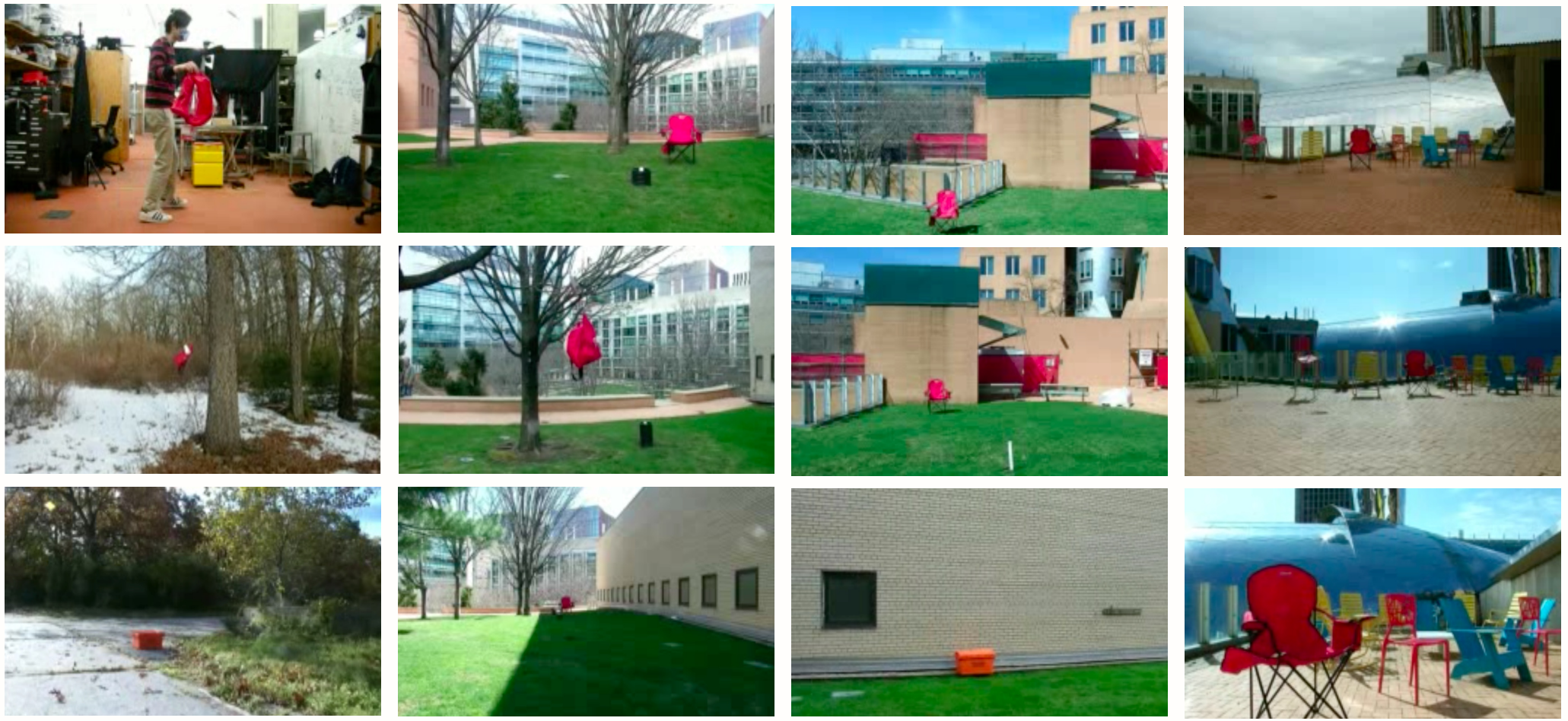


Liquid CfC = liquid closed-form continuous-time
Neural ODE = neural ordinary differential equations
LSTM = long short-term memory

Chahine* & Hasani* et al. Science Robotics 2023

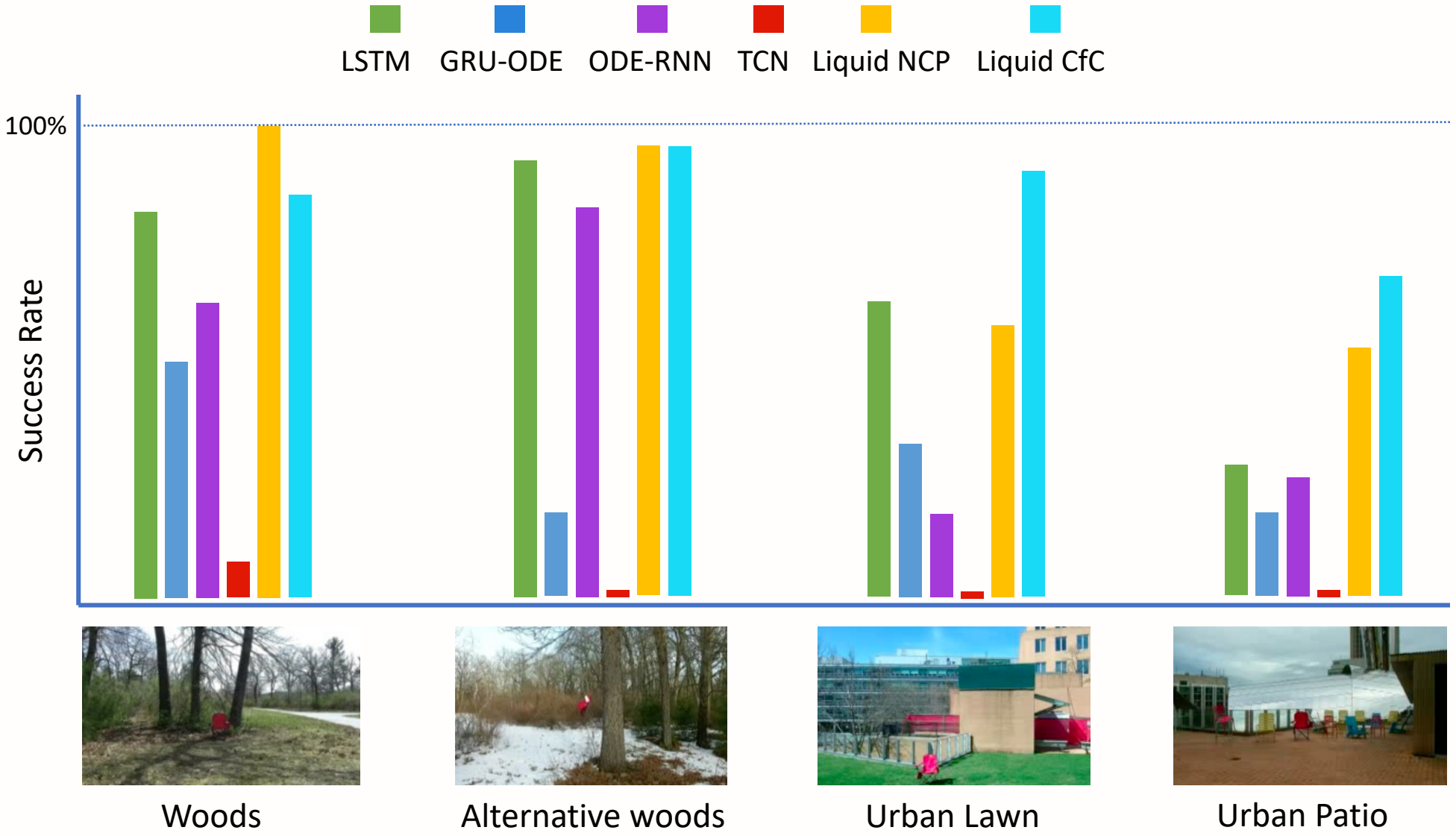
Liquid networks zero-shot transfer performance.

Generalization out-of-distribution



Chahine* & Hasani* et al. Science Robotics 2023

In and Out of Distribution Performance



Chahine* & Hasani* et al. Science Robotics 2023

Science Robotics
AAAS

Science Robotics
AAAS

Liquid Structural State-Space Models

Liquid S4

$$\dot{x}(t) = \left[\mathbf{A} + \mathbf{B}\, u(t)\right] x(t) + \mathbf{B}\, u(t)$$
$$y(t) = \mathbf{C}\, x(t)$$

$x(t)$ is an N-dimensional latent state, receiving a 1-dimensional input signal $u(t)$, and computing a 1-dimensional output signal $y(t)$. $\mathbf{A}^{(N\times N)}$, $\mathbf{B}^{(N\times 1)}$, and $\mathbf{C}^{(1\times N)}$. Note that $\mathbf{D}$ is set to zero for simplicity.

Hasani et al. ICLR 2023

Liquid vs Transformers

Performance on sequence modeling tasks:

- *Sequential image processing*

  46% accuracy improvement over Transformers

- *Text sentiment analysis*

  10% accuracy improvement over Transformers

- *Recovering long info paths from 2D trajectories*

  16% accuracy improvement (context length 1k)
  96% improvement (context length 16k)

- *Log list operations*

  30% accuracy improvement over Transformers

- *Information retrieval*

  11% accuracy improvement

Model (input length)	ListOps 2048	IMDB 2048	AAN 4000	CIFAR 1024	Pathfinder 1024	Path-X 16384	Avg.
Random*	10.00	50.00	50.00	10.00	50.00	50.00	36.67
Transformer* (Vaswani et al., 2017)	36.37	64.27	57.46	42.44	71.40	x	54.39
Local Att.* (Tay et al., 2020b)	15.82	52.98	53.39	41.46	66.63	x	46.06
Sparse Transformer* (Child et al., 2019)	17.07	63.58	59.59	44.24	71.71	x	51.24
Longformer* (Beltagy et al., 2020)	35.63	62.85	56.89	42.22	69.71	x	53.46
Linformer* (Wang et al., 2020)	16.13	65.90	53.09	42.34	75.30	x	50.55
Reformer* (Kitaev et al., 2019)	37.27	56.10	53.40	38.07	68.50	x	50.56
Sinkhorn Trans.* (Tay et al., 2020a)	33.67	61.20	53.83	41.23	67.45	x	51.23
BigBird* (Zaheer et al., 2020)	36.05	64.02	59.29	40.83	74.87	x	55.01
Linear Trans.* (Katharopoulos et al., 2020)	16.13	65.90	53.09	42.34	75.30	x	50.46
Performer* (Choromanski et al., 2020)	18.01	65.40	53.82	42.77	77.05	x	51.18
FNet** (Lee-Thorp et al., 2021)	35.33	65.11	59.61	38.67	77.80	x	54.42
Nyströmformer** (Xiong et al., 2021)	37.15	65.52	79.56	41.58	70.94	x	57.46
Luna-256** Ma et al. (2021)	37.25	64.57	79.29	47.38	77.72	x	59.37
H-Transformer-1D** (Zhu and Soricut, 2021)	49.53	78.69	63.99	46.05	68.78	x	61.41
CDIL (Cheng et al., 2022)	44.05	86.78	85.36	66.91	91.70	x	74.96
DSS (Gupta, 2022)	57.6	76.6	87.6	85.8	84.1	85.0	79.45
S4 (original)** (Gu et al., 2022a)	58.35	76.02	87.09	87.26	86.05	88.10	80.48
S4-LegS (Gu et al., 2022b)	59.60 (0.07)	86.82 (0.13)	90.90 (0.15)	88.65 (0.23)	94.20 (0.25)	96.35	86.09
S4-FouT (Gu et al., 2022b)	57.88 (1.90)	86.34 (0.31)	89.66 (0.88)	89.07 (0.19)	94.46 (0.26)	x	77.90
S4-LegS/FouT (Gu et al., 2022c)	60.45 (0.75)	86.78 (0.26)	90.30 (0.28)	89.00 (0.26)	94.44 (0.08)	x	78.50
S4D-LegS (Gu et al., 2022b)	60.47 (0.34)	86.18 (0.43)	89.46 (0.14)	88.19 (0.26)	93.06 (1.24)	91.95	84.89
S4D-Inv (Gu et al., 2022b)	60.18 (0.35)	87.34 (0.20)	91.09 (0.01)	87.83 (0.37)	93.78 (0.25)	92.80	85.50
S4D-Lin (Gu et al., 2022b)	60.52 (0.51)	86.97 (0.23)	90.96 (0.09)	87.93 (0.34)	93.96 (0.60)	x	78.39
S5 (Smith et al., 2022)	61.00	86.51	88.26	86.14	87.57	85.25	82.46
Liquid-S4 (ours)	**62.75** (0.2) p = 5	**89.02** (0.04) p=6	**91.20** (0.01) p=2	**89.50** (0.4) p=3	**94.8** (0.2) p=2	**96.66**(0.001) p=2	**87.32**

Hasani et al. ICLR 2023

Liquid vs Transformers

Performance on sequential long-term signal classification:

30%

Performance improvement over Transformers

Model	Accuracy
Transformer (Trinh et al., 2018)	62.2
FlexConv (Romero et al., 2021a)	80.82
TrellisNet (Bai et al., 2018)	73.42
LSTM	63.01
r-LSTM (Trinh et al., 2018)	72.2
UR-GRU (Gu et al., 2020b)	74.4
HiPPO-RNN (Gu et al., 2020a)	61.1
LipschitzRNN (Erichson et al., 2021)	64.2
Liquid-S4-PB (ours)	**92.02** (0.14) p=3

Hasani et al. ICLR 2023

Use Case: Order Book Price Forecasting – Overview

Typical LOBSTER Data Sample

$$\underbrace{[p_1^a, v_1^a, p_1^b, v_1^b, \ldots, p_d^a, v_d^a, p_d^b, v_d^b]}_{\text{Length } 4d \ (d=\text{depth})}$$

$p_d^a,\ p_d^b$ - Ask/Bid prices at depth d

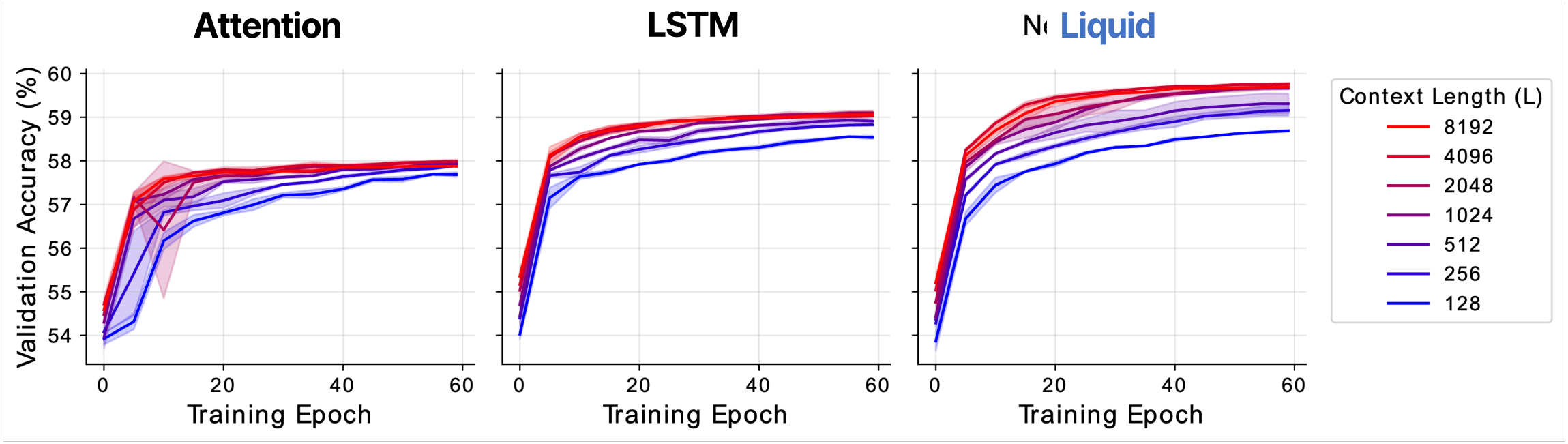
$v_d^a,\ v_d^b$ - Ask/Bid volumes at depth d

LOBSTER Dataset

- ✓ High frequency order book data at irregularly frequencies
- ✓ Non-stationary with low signal-to-noise ratio
- ✓ Signals at multiple timescales, with long-range interactions
- ✓ **Goal: Predict whether mid price will move up/down in the next t_f seconds**

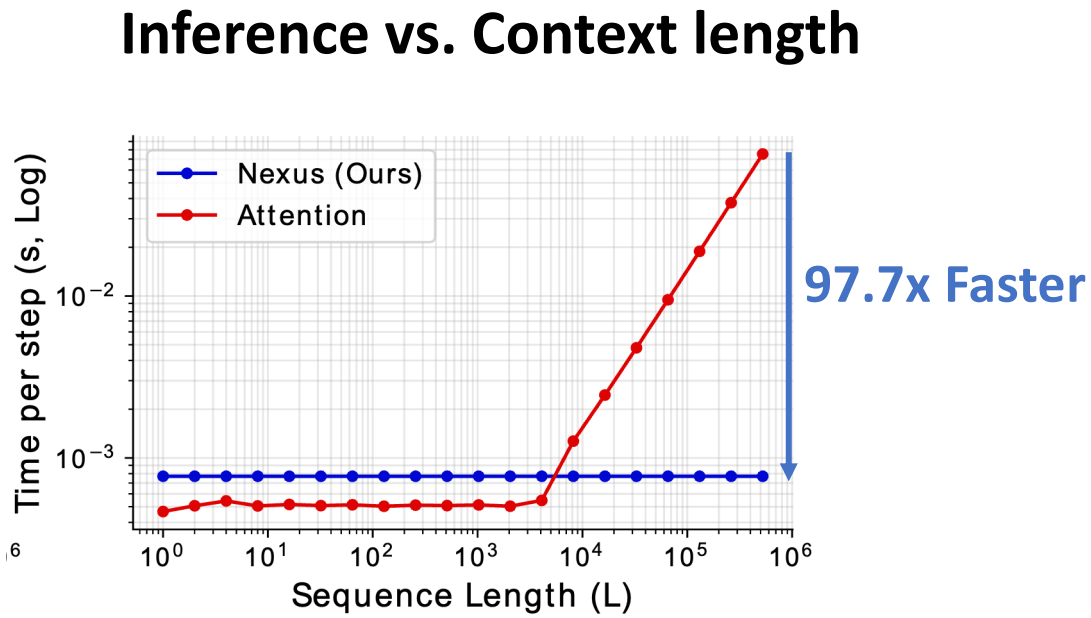
Use Case: Order Book Price Forecasting – Results

Context Length vs. Accuracy



Liquid AI's models can effectively leverage long contexts lengths

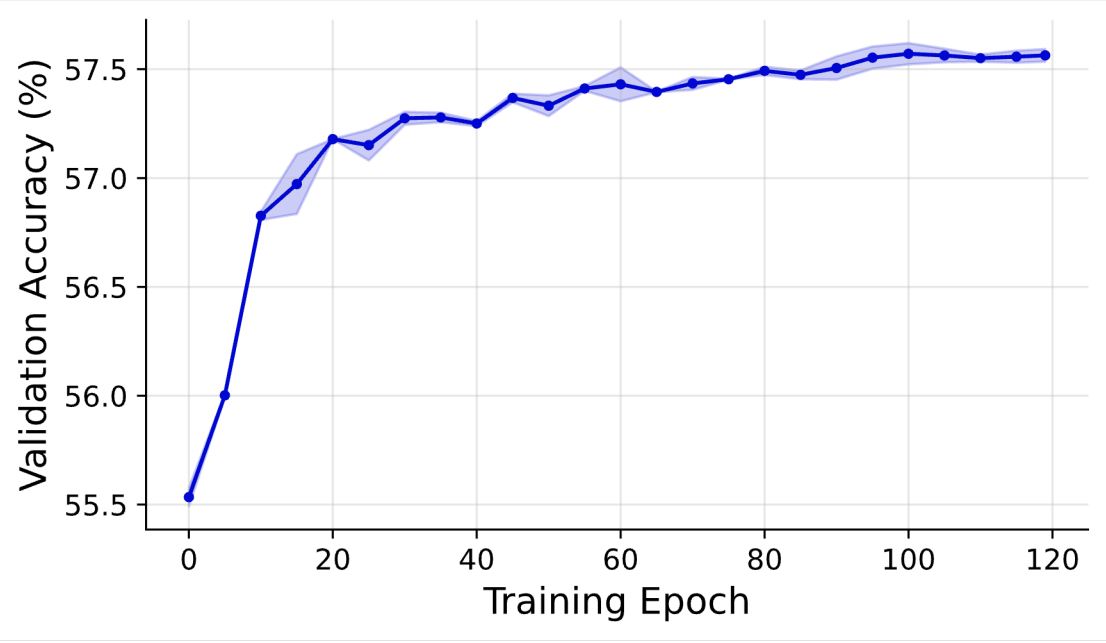
Use Case: Order Book Price Forecasting – Results



At inference time, Liquid AI models' inference speeds are Constant in sequence length, while attention gets slower with longer sequences

Use Case: Order Book Price Forecasting – Results

GPSM Nexus model trained on 130k context length with $\Delta t = 0.05s$



Liquid AI models scale to ultra long (>130k) context lengths *in training*

Use Case: Credit Card Fraud Detection

Liquid resulted in **5% recall accuracy improvement** and **82% reduction in false positive rate**

	LSTM-Attention	Liquid GPSM	
Parameters	32k	7k	34k
Recall (%)	94.9%	99.4%	**99.9%**
False Positive	0.027	0.01	**0.002**
Training Step (ms)	4.52	4.36	8.09
Inference (ms)	1.89	**1.35**	2.48

Liquid vs Transformers

Performance on medical time-series (Human vital signals - BIDMC)

33x

Accuracy improvement over Transformers.

	BIDMC		
Model	HR	RR	SPO2
UnICORNN (Rusch & Mishra, 2021b)	1.39	1.06	0.869
coRNN (Rusch & Mishra, 2021a)	1.81	1.45	-
CKConv*	2.05	1.214	1.051
NRDE (Morrill et al., 2021)	2.97	1.49	1.29
LSTM (Rusch & Mishra, 2021b)	10.7	2.28	-
Transformer*	12.2	2.61	3.02
XGBoost (Tan et al., 2021)	4.72	1.67	1.52
Random Forest (Tan et al., 2021)	5.69	1.85	1.74
Ridge Regress. (Tan et al., 2021)	17.3	3.86	4.16
Liquid-S4-PB (ours)	**0.303** (0.002) p=3	**0.158** (0.001) p=2	**0.066** (0.002) p=4

Hasani et al. ICLR 2023

Use Case: Speech Recognition on the Edge

	Attention	Liquid Nexus Model
Hidden Size	128	128
Test Accuracy (5 epochs)	0.86	**0.943**
Time per Batch (s)	158.71	**0.4**

- Controlling devices with voice commands is challenging:
  - Requires an efficient & accurate model that recognizes words amidst noisy / low-quality audio

- **Liquid Nexus Model efficiently and accurately classifies high sample rate audio data in Speech Commands.**

- **Over 300X Time per Batch improvement** while maintaining efficiency

USE CASE

Liquid Networks to fly US Air Force jet aircraft X-62 VISTA (derived from the F-16D Fighting Falcon)

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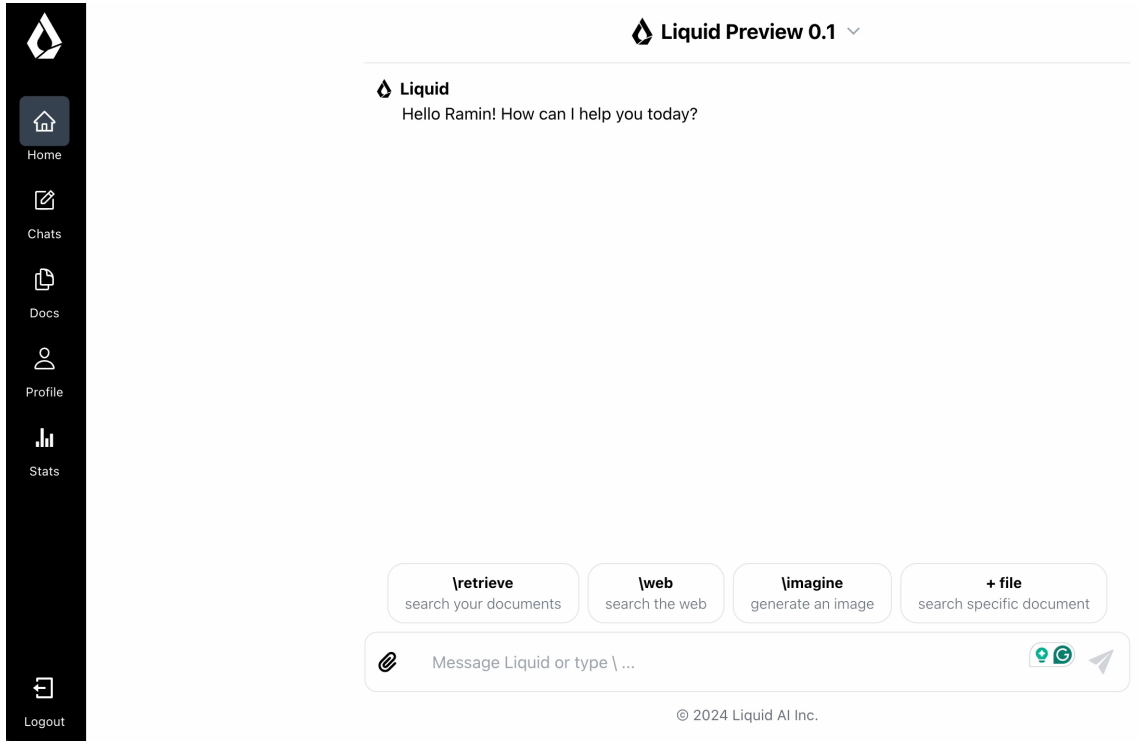
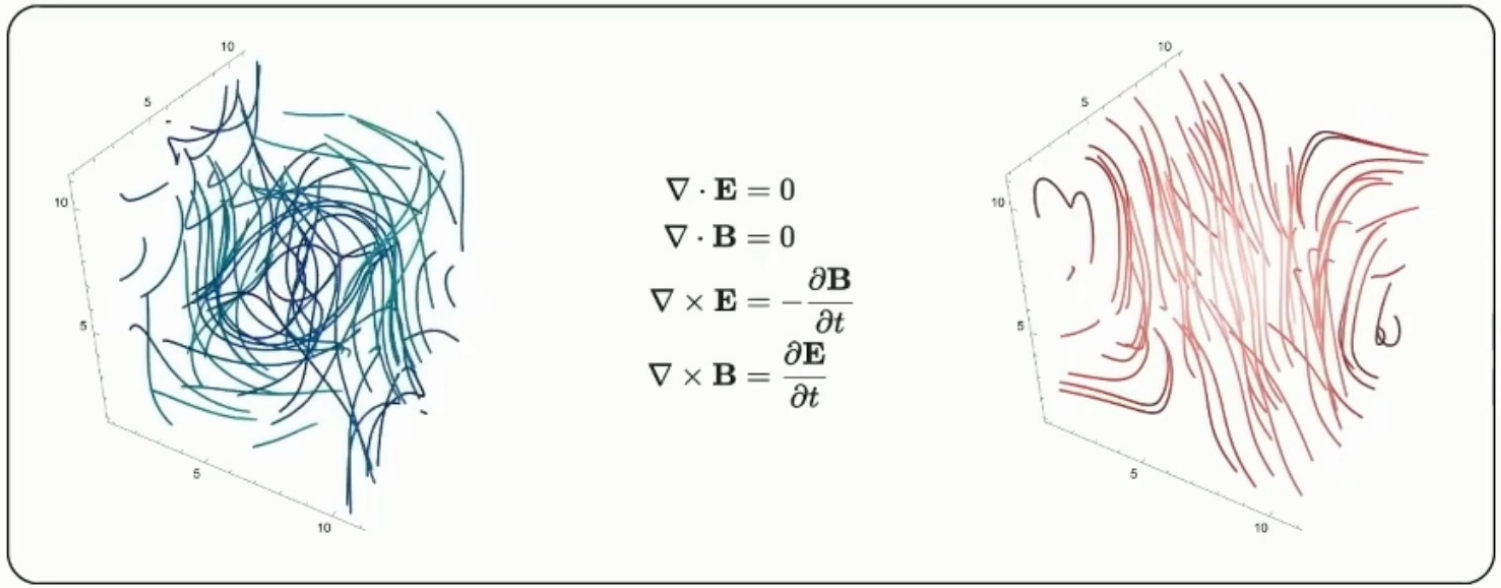
Creator: Hand-out | Photo Credit: Lockheed Martin Aeronautics

Liquid Foundation Models (LFMs)

Model Sizes	Context Length
1-100B	32k-20M

Scaled Liquid models from 80k-parameter network that drive cars to:





GP PRIORS FOR SYSTEMS OF LINEAR PDE WITH CONSTANT COEFFICIENTS

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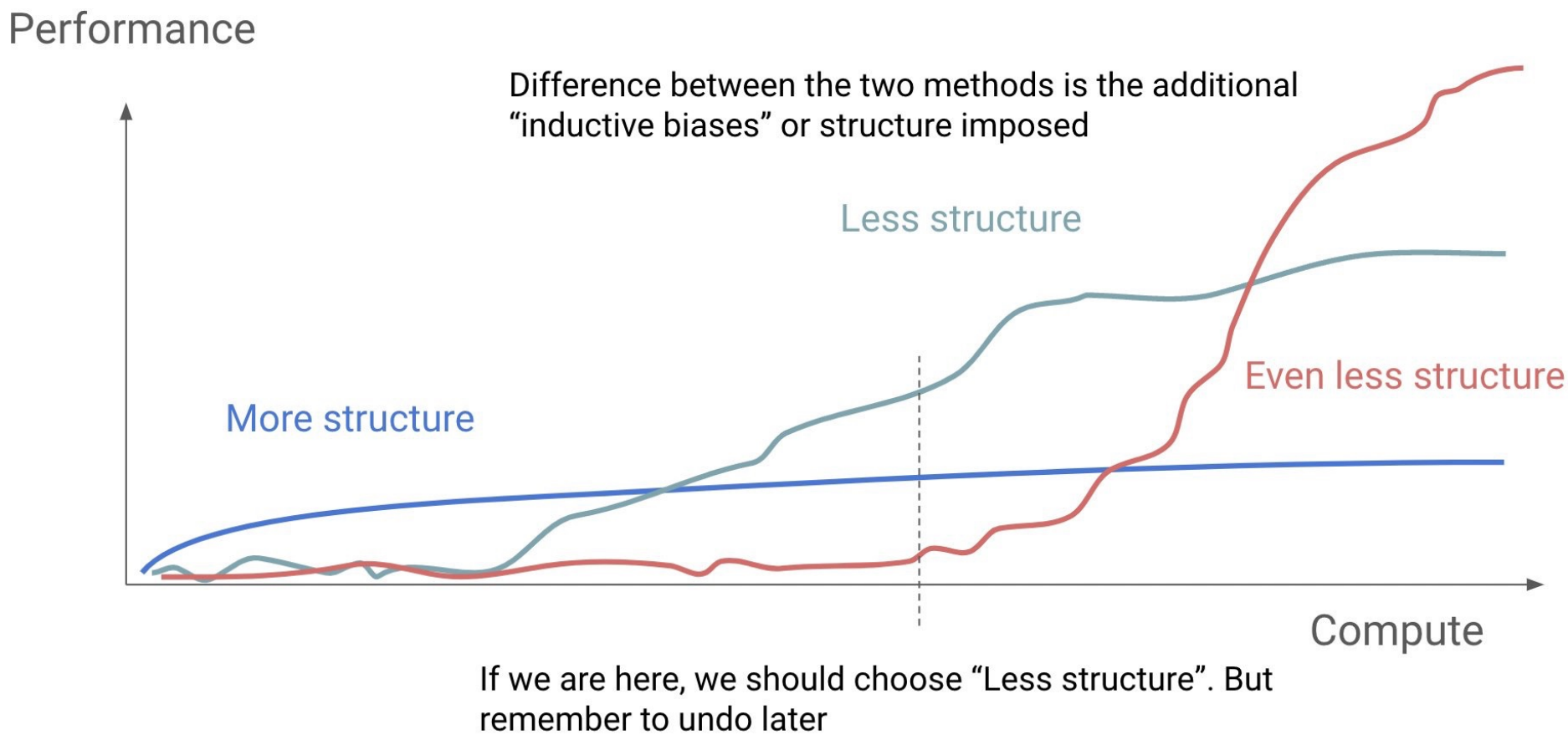

Credit: Hyung Won Chung, OpenAI
Talk at Stanford CS 25, yesterday!

Resources

Get Hands-on with **LTC-based networks:**
github.com/mlech26l/keras-ncp
github.com/raminmh/liquid_time_constant_networks

Get Hands-on with the **closed-form networks:**
github.com/raminmh/CfC

Get Hands-on with **Liquid-S4:**
https://github.com/raminmh/liquid-s4

Get in touch: rhasani@mit.edu

Learning representations from sequences of data requires expressive temporal and structural credit assignment.

Liquid Time-Constant (LTC) networks

1. Linear state-space model

$$d\mathbf{x}(t)/dt = -\mathbf{x}(t)/\tau + \mathbf{S}(t) \qquad \mathbf{S}(t) \in \mathbb{R}^M$$

2. Non-linear synapse Model

$$\mathbf{S}(t) = f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)(A - \mathbf{x}(t))$$

$$\frac{d\mathbf{x}(t)}{dt} = -\left[\frac{1}{\tau} + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)\right]\mathbf{x}(t) + f(\mathbf{x}(t), \mathbf{I}(t), t, \theta)A$$

"Liquid" = variable

Liquid Networks Limitations

1. Liquid networks' complexity is tied to their numerical solver
2. Learning Long term dependencies due to gradient issues

Solutions

1. Closed-form Continuous-Time Neural Networks [Hasani et al. Nature MI, 2022]

2. Mixed Memory Recurrent Networks [Lechner and Hasani, MemARI NeurIPS 2022]

3. **Bring Liquid Dynamics in Structural State Space Models** [Hasani et al. ICLR 2023]

Structural State-Space Models [Gu et al. ICLR 2022]

State-Space Models (SSM)

State-Space Models

$$\dot{x}(t) = \mathbf{A}\, x(t) + \mathbf{B}\, u(t)$$
$$y(t) = \mathbf{C}\, x(t) + \mathbf{D}\, u(t)$$

$$\mathbf{A}^{(N\times N)},\ \mathbf{B}^{(N\times 1)},\ \mathbf{C}^{(1\times N)} \text{ and } \mathbf{D}^{(1\times 1)}$$

$x(t)$ is an N-dimensional latent state, receiving a 1-dimensional input signal $u(t)$

Discretization

$$x_k = \overline{\mathbf{A}}\, x_{k-1} + \overline{\mathbf{B}}\, u_k$$
$$y_k = \overline{\mathbf{C}}\, x_k$$

$$\overline{\mathbf{A}} = (\mathbf{I} - \frac{\delta t}{2}\mathbf{A})^{-1}(\mathbf{I} + \frac{\delta t}{2}\mathbf{A})$$
$$\overline{\mathbf{B}} = (\mathbf{I} - \frac{\delta t}{2}\mathbf{A})^{-1}\, \delta t\, \mathbf{B}$$
$$\overline{\mathbf{C}} = \mathbf{C}$$

Structural State-Space Models

SSM Convolutional Kernel

$$x_k = \overline{\mathbf{A}}\, x_{k-1} + \overline{\mathbf{B}}\, u_k$$
$$y_k = \overline{\mathbf{C}}\, x_k$$

$$x_0 = \overline{\mathbf{B}}u_0, \quad x_1 = \overline{\mathbf{AB}}u_0 + \overline{\mathbf{B}}u_1, \quad x_2 = \overline{\mathbf{A}}^2\overline{\mathbf{B}}u_0 + \overline{\mathbf{AB}}u_1 + \overline{\mathbf{B}}u_2, \quad \dots$$
$$y_0 = \overline{\mathbf{CB}}u_0, \quad y_1 = \overline{\mathbf{CAB}}u_0 + \overline{\mathbf{CB}}u_1, \quad y_2 = \overline{\mathbf{CA}}^2\overline{\mathbf{B}}u_0 + \overline{\mathbf{CAB}}u_1 + \overline{\mathbf{CB}}u_2, \quad \dots$$

The mapping $u_k \to y_k$ can now can be formulated into a convolutional kernel explicitly:

$$y_k = \overline{\mathbf{CA}}^k\overline{\mathbf{B}}u_0 + \overline{\mathbf{CA}}^{k-1}\overline{\mathbf{B}}u_1 + \dots \overline{\mathbf{CAB}}u_{k-1} + \overline{\mathbf{CB}}u_k$$
$$y = \overline{\mathbf{K}} * u$$

$$\overline{\mathbf{K}} \in \mathbb{R}^L := \mathcal{K}_L(\overline{\mathbf{C}}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) := \left(\overline{\mathbf{CA}}^i\overline{\mathbf{B}}\right)_{i\in[L]} = \left(\overline{\mathbf{CB}}, \overline{\mathbf{CAB}}, \dots, \overline{\mathbf{CA}}^{L-1}\overline{\mathbf{B}}\right)$$

Diagonal Plus Low Rank decomposition (DPLR) + Hippo Initialization will do the job!

$$\boldsymbol{A}_{nk} = -\begin{cases} (2n+1)^{\frac{1}{2}}(2k+1)^{\frac{1}{2}} & n > k \\ n+1 & n = k \\ 0 & n < k \end{cases}$$

$$\boldsymbol{B}_n = (2n+1)^{\frac{1}{2}} \quad \boldsymbol{P}_n = (n+1/2)^{\frac{1}{2}}$$

(HiPPO-LegS matrix used in S4)

(4)

$$\boldsymbol{A}^{(N)}_{nk} = -\begin{cases} (n+\frac{1}{2})^{1/2}(k+\frac{1}{2})^{1/2} & n > k \\ \frac{1}{2} & n = k \\ (n+\frac{1}{2})^{1/2}(k+\frac{1}{2})^{1/2} & n < k \end{cases}$$

$$\boldsymbol{A} = \boldsymbol{A}^{(N)} - \boldsymbol{P}\boldsymbol{P}^\top, \qquad \boldsymbol{A}^{(D)} := \mathrm{eig}(\boldsymbol{A}^{(N)})$$

(Normal / diagonal plus low-rank form)

Gu et al. ICLR 2022

Liquid Structural State-Space Models

Liquid S4

$$\dot{x}(t) = \left[\mathbf{A} + \mathbf{B}\, u(t)\right] x(t) + \mathbf{B}\, u(t)$$
$$y(t) = \mathbf{C}\, x(t)$$

$x(t)$ is an N-dimensional latent state, receiving a 1-dimensional input signal $u(t)$, and computing a 1-dimensional output signal $y(t)$. $\mathbf{A}^{(N\times N)}$, $\mathbf{B}^{(N\times 1)}$, and $\mathbf{C}^{(1\times N)}$. Note that $\mathbf{D}$ is set to zero for simplicity.

Hasani et al. ICLR 2023

Liquid Structural State-Space Models

Liquid S4

A forward Euler Transformation of the Linear LTC

$$x_k = \left(\overline{\mathbf{A}} + \overline{\mathbf{B}}\, u_k\right) x_{k-1} + \overline{\mathbf{B}}\, u_k$$
$$y_k = \overline{\mathbf{C}}\, x_k$$

$\overline{\mathbf{A}} = \mathbf{I} + \frac{\delta t}{2}\mathbf{A}$, $\overline{\mathbf{B}} = \delta t\, \mathbf{B}$, and $\overline{\mathbf{C}} = \mathbf{C}$

$$x_0 = \overline{\mathbf{B}} u_0,$$
$$y_0 = \overline{\mathbf{CB}} u_0$$
$$x_1 = \overline{\mathbf{AB}} u_0 + \overline{\mathbf{B}} u_1 + \overline{\mathbf{B}}^2 u_0 u_1,$$
$$y_1 = \overline{\mathbf{CAB}} u_0 + \overline{\mathbf{CB}} u_1 + \overline{\mathbf{CB}}^2 u_0 u_1$$
$$x_2 = \overline{\mathbf{A}}^2\overline{\mathbf{B}} u_0 + \overline{\mathbf{AB}} u_1 + \overline{\mathbf{B}} u_2 + \overline{\mathbf{AB}}^2 u_0 u_1 + \overline{\mathbf{AB}}^2 u_0 u_2 + \overline{\mathbf{B}}^2 u_1 u_2 + \overline{\mathbf{B}}^3 u_0 u_1 u_2$$
$$y_2 = \overline{\mathbf{CA}}^2\overline{\mathbf{B}} u_0 + \overline{\mathbf{CAB}} u_1 + \overline{\mathbf{CB}} u_2 + \overline{\mathbf{CAB}}^2 u_0 u_1 + \overline{\mathbf{CAB}}^2 u_0 u_2 + \overline{\mathbf{CB}}^2 u_1 u_2 + \overline{\mathbf{CB}}^3 u_0 u_1 u_2$$
$$\dots$$

$$y_k = \overline{\mathbf{CA}}^k\overline{\mathbf{B}} u_0 + \overline{\mathbf{CA}}^{k-1}\overline{\mathbf{B}} u_1 + \dots \overline{\mathbf{CAB}} u_{k-1} + \overline{\mathbf{CB}} u_k + \sum_{p=2}^{\mathcal{P}} \sum_{u_i u_{i+1} \dots u_p \in \Pi(k+1,p)} \overline{\mathbf{CA}}^{(k+1-p-i)}\overline{\mathbf{B}}^p u_i u_{i+1} \dots u_p$$

$$\text{for } i \in \mathbb{Z} \text{ and } i \geq 0, \quad \rightarrow \quad y = \overline{\mathbf{K}} * u + \overline{\mathbf{K}}_{\text{liquid}} * u_{\text{correlations}}$$

Hasani et al. ICLR 2023

Liquid Structural State-Space Models

Example: Liquid S4

$$\overline{\mathbf{K}}_{\text{liquid}} * u_{\text{correlations}} = \begin{pmatrix} \overline{\mathbf{CA}}^{(k-1)}\overline{\mathbf{B}}^{2}, \\ \vdots \\ \overline{\mathbf{CB}}^{2}, \\ \vdots \\ \overline{\mathbf{CA}}^{(k-2)}\overline{\mathbf{B}}^{3}, \\ \vdots \\ \overline{\mathbf{CB}}^{3}, \\ \vdots \\ \overline{\mathbf{CA}}^{(k-3)}\overline{\mathbf{B}}^{4}, \\ \vdots \\ \overline{\mathbf{CB}}^{4} \end{pmatrix}^{T} * \begin{bmatrix} u_0 u_1 \\ \vdots \\ u_{k-1}u_k \\ \vdots \\ u_0 u_1 u_2 \\ \vdots \\ u_{k-2}u_{k-1}u_k \\ \vdots \\ u_0 u_1 u_2 u_3 \\ \vdots \\ u_{k-3}u_{k-2}u_{k-1}u_k \end{bmatrix}$$

Here, $u_{\text{correlations}}$ is a vector of length $\binom{k+1}{2} + \binom{k+1}{3} + \binom{k+1}{4}$, and the kernel $\overline{\mathbf{K}}_{\text{liquid}} \in \mathbb{R}^{\binom{k+1}{2}+\binom{k+1}{3}+\binom{k+1}{4}}$. This additional kernel takes the temporal correlation of incoming input samples into consideration.

Hasani et al. ICLR 2023

Liquid Structural State-Space Models

Liquid S4 Kernel

$$y_k = \overline{\mathbf{CA}}^k \overline{\mathbf{B}} u_0 + \overline{\mathbf{CA}}^{k-1} \overline{\mathbf{B}} u_1 + \ldots \overline{\mathbf{CAB}} u_{k-1} + \overline{\mathbf{CB}} u_k +$$

$$\sum_{p=2}^{\mathcal{P}} \sum_{u_i u_{i+1} \ \ldots \ u_p \in \Pi(k+1,p)} \overline{\mathbf{CA}}^{(k+1-p-i)} \overline{\mathbf{B}}^p u_i u_{i+1} \ \ldots \ u_p$$

$$\text{for } i \in \mathbb{Z} \text{ and } i \geq 0, \quad \rightarrow \quad y = \overline{\mathbf{K}} * u + \overline{\mathbf{K}}_{\text{liquid}} * u_{\text{correlations}}$$

$$\overline{\mathbf{K}}_{\text{liquid}} \in \mathbb{R}^{\tilde{L}} := \mathcal{K}_L(\overline{\mathbf{C}}, \overline{\mathbf{A}}, \overline{\mathbf{B}}) := \left(\overline{\mathbf{CA}}^{(\tilde{L}-i-p)} \overline{\mathbf{B}}^p\right)_{i \in [\tilde{L}],\ p \in [\mathcal{P}]} = \left(\overline{\mathbf{CA}}^{\tilde{L}-2} \overline{\mathbf{B}}^2, \ldots, \overline{\mathbf{CB}}^p\right)$$

Hasani et al. ICLR 2023

Liquid Structural State-Space Models

Liquid-S4 Kernel

Algorithm 1 LIQUID-S4 KERNEL - The S4 convolution kernel (highlighted in black) is used from Gu et al. (2022a) and Gu et al. (2022b). Liquid kernel computation is highlighted in purple.

Input: S4 parameters $\boldsymbol{\Lambda}, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{C}^N$, step size Δ, liquid kernel order $\mathcal{P}$, inputs seq length L, liquid kernel sequence length $\tilde{L}$

Output: SSM convolution kernel $\overline{\boldsymbol{K}} = \mathcal{K}_L(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{C}})$ and SSM liquid kernel $\overline{\boldsymbol{K}}_{liquid} = \mathcal{K}_{\tilde{L}}(\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}, \overline{\boldsymbol{C}})$ for $\boldsymbol{A} = \boldsymbol{\Lambda} - \boldsymbol{P}\boldsymbol{P}^*$ (Eq. 6)

$\widetilde{\boldsymbol{C}} \leftarrow \left(\boldsymbol{I} - \overline{\boldsymbol{A}}^L\right)^* \overline{\boldsymbol{C}}$ ▷ Truncate SSM generating function (SSMGF) to length L

$\begin{bmatrix} k_{00}(\omega) & k_{01}(\omega) \\ k_{10}(\omega) & k_{11}(\omega) \end{bmatrix} \leftarrow \left[\widetilde{\boldsymbol{C}}\ \boldsymbol{P}\right]^* \left(\frac{2}{\Delta}\frac{1-\omega}{1+\omega} - \boldsymbol{\Lambda}\right)^{-1} [\boldsymbol{B}\ \boldsymbol{P}]$ ▷ Black-box Cauchy kernel

$\hat{\boldsymbol{K}}(\omega) \leftarrow \frac{2}{1+\omega}\left[k_{00}(\omega) - k_{01}(\omega)(1+k_{11}(\omega))^{-1}k_{10}(\omega)\right]$ ▷ Woodbury Identity

$\hat{\boldsymbol{K}} = \{\hat{\boldsymbol{K}}(\omega) : \omega = \exp(2\pi i \frac{k}{L})\}$ ▷ Evaluate SSMGF at all roots of unity $\omega \in \Omega_L$

$\overline{\boldsymbol{K}} \leftarrow \mathsf{iFFT}(\hat{\boldsymbol{K}})$ ▷ Inverse Fourier Transform

if Mode == KB **then** ▷ Liquid-S4 Kernel as shown in Eq. 14

  for p in $\{2, \ldots, \mathcal{P}\}$ **do**

    $\overline{\boldsymbol{K}}_{\text{liquid}=p} = \left[\overline{\boldsymbol{K}}_{(L-\tilde{L},L)} \odot \overline{\boldsymbol{B}}^{p-1}_{(L-\tilde{L},L)}\right] * \mathbf{J}_{\tilde{L}}$ ▷ $\mathbf{J}_{\tilde{L}}$ is a backward identity matrix

    $\overline{\boldsymbol{K}}_{\text{liquid}}.\text{append}(\overline{\boldsymbol{K}}_{\text{liquid}=p})$

  end for

else if Mode == PB **then** ▷ Liquid-S4 Kernel of Eq. 14 with $\overline{\boldsymbol{A}}$ reduced to Identity.

  for p in $\{2, \ldots, \mathcal{P}\}$ **do**

    $\overline{\boldsymbol{K}}_{\text{liquid}=p} = \overline{\boldsymbol{C}} \odot \overline{\boldsymbol{B}}^{p-1}_{(L-\tilde{L},L)}$

    $\overline{\boldsymbol{K}}_{\text{liquid}}.\text{append}(\overline{\boldsymbol{K}}_{\text{liquid}=p})$

  end for

end if